INVERSION THEOREM ASSOCIATED WITH LINEAR CANONICAL-MELLIN TRANSFORM

Vidya Sharma¹ and Nitin Dhange²

¹Department of Mathematics, Smt. Narsamma Arts, Commerce & Science College Amravati, (M.S.) India.
²Department of Mathematics, Bhawabhuti Mahavidyalaya, Amgaon, (M.S.) India.

Communicated: 10.12.2022
Revision: 20.01.2023 & 24.01.2023
Accepted: 26.01.2023
Published: 30.01.2023

ABSTRACT:
The Linear canonical-Mellin transform is a mixed type of integral transform in which function is both linear canonical and Mellin transformable. Extension of some transformations to generalized functions have been done time to time and their properties have been studied by various mathematicians. However, there is much scope in extending double transformations to a certain class of generalized functions. In this paper, Linear Canonical-Mellin transform is extended in the distributional generalized sense and inversion theorem for Linear Canonical-Mellin transform is proved which can be used to retrieve original function to be transformed.

Keywords: - Inversion theorem, Linear canonical transform, Mellin transform, Testing function space.

INTRODUCTION:
Linear canonical-Mellin transform is a mixed type of integral transform with composition of linear canonical and Mellin transform in which linear canonical transform (LCT) is an integral transform with generalized kernel and Mellin transform is a basic integral transform. The LCT is a four-parameter family of integral transform defined by [1]:

\[ L_a(b)[\phi(u)] = \Phi(u) = \int_{-\infty}^{\infty} \phi(t) K_a(u, t) dt, \quad b \neq 0 \]

\[ = \frac{1}{\sqrt{2\pi b}} e^{-\frac{x^2}{2b}} \frac{1}{\sqrt{2\pi x}} \frac{1}{\sqrt{2\pi x}} \quad \text{where the LCT kernel } K_a(u, t) \text{ is given by the operator } \]

\[ K_a(u, t) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{t^2}{2b}} \frac{1}{\sqrt{2\pi x}} \frac{1}{\sqrt{2\pi x}} \]

and parameters \( a, b, c, d \) are real numbers satisfying \( ad - bc = 1 \). On condition that the parameters satisfy \( b = 0 \), the LCT is essentially a scaling and chirp multiplication operations. Without loss of generality, we therefore focus mainly on the LCT in the case of \( b \neq 0 \). In that case, the inverse LCT is

\[ \phi(t) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \Phi(u) e^{\frac{t^2}{2b}} du. \]

It is easy to verify that the LCT with parameters \( (a, b, c, d) = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta) \) reduces to the fractional Fourier transform (FRFT), which, in the specific case \( \theta = \frac{\pi}{4} \), becomes the Fourier transform and Parseval relation for LCT is given by [2]:

\[ \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt = \int_{-\infty}^{\infty} F(u) \overline{G(u)} du \]

The Mellin transform is defined as [3]:

\[ M[f; s] \equiv F(s) = \int_{0}^{\infty} f(x) x^{s-1} dx \]

and its Parseval relation is given by

\[ \int_{0}^{\infty} f(x) \overline{g(x)} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(1 - s) \overline{G(s)} ds \]

Chii-Huei Yu [4] provided a new technique to determine some definite integrals using Parseval’s theorem and this technique can be applied to solve another definite integral problems. Soo-Chang Pei [5,6] derived many important properties of discrete fractional

http://doi.org/10.29369/ijrbat.2023.010.1.0057
Fourier transform and discussed some applications, such as the filter design and pattern recognition Zemanian [7] studied several integral transforms in the distributional generalized sense. Sharma et. al. [8, 9] had generalized many integral transforms to the distribution of compact support and provided some operational properties of two-dimensional fractional Mellin transform, two dimensional fractional Fourier-Mellin transform. The aim of this paper is to prove inversion formula of Linear Canonical-Mellin transform and Parseval’s theorem.

**Linear Canonical-Mellin Transform (LCMT)**

**Definition:** The conventional Linear Canonical-Mellin transform is defined as follows:

\[
L_{A}M(f(t,x)) = F^{A}M(u,s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,x) K(t,u,s) dt \ dx
\]

where 

\[
K(t,u,s) = \frac{1}{\sqrt{2\pi i b}} e^{\frac{iu^{2}-b^{2}us}{2}} \ t^{s^{-1}} , \quad b \neq 0, s > 0.
\]

**The Testing Function Space \( E(R^n) \)**

An infinitely differentiable complex valued smooth function \( \varphi \) on \( R^n \) belongs to \( E(R^n) \), if for each compact set \( K \subset S_a, \ i \subset S_b \),

where 

\( S_a = \{ t: t \in R^n, |t| \leq a, a > 0 \} \) and \( S_b = \{ x: x \in R^n, |x| \leq b, b > 0 \}, K, J \in R^n \), 

\( \gamma_{E,Lq} = \sup_{x \in \Omega} |D^{l}_{x}D^{q}_{s} \varphi(t,x)| < \infty , \quad l, q = 0,1,2, \ldots \)

Thus \( E(R^n) \) will denote the space of all \( \varphi \in E(R^n) \) with support contained in \( S_a \) and \( S_b \). Moreover, we say that \( f \) is a linear canonical-Mellin transformable if it is a member of \( E^{*} \), the dual space of \( E \).

**Distributional Generalized Linear Canonical-Mellin Transform**

The distributional Linear Canonical-Mellin transform of \( f(t,x) \in E^{*}(R^n) \) is defined by

\[
L_{A}M(f(t,x)) = F^{A}M(u,s) = (f(t,x), K(t,u,s))
\]

(2.1)

where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( ad - bc = 1 \) and

\[
K(t,u,s) = \frac{1}{\sqrt{2\pi i b}} e^{\frac{iu^{2}-b^{2}us}{2}} \ t^{s^{-1}} , \quad b \neq 0, s > 0.
\]

The right-hand side of (1) is meaningful because \( K(t,x,u,s) \in E \) and \( f(t,x) \in E^{*} \).

**Inversion Formula:**

If Linear Canonical-Mellin transform of \( f(t,x) \) is given by

\[
L_{A}M(f(t,x)) = F^{A}M(u,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,x) e^{\frac{i(a^{2}-2au+du^{2})}{2} x^{s^{-1}} dt \ dx
\]

Then its inverse \( f(t,x) \) is given by

\[
f(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{A}M(u,s) e^{-\frac{i(a^{2}-2au+du^{2})}{2} x^{s^{-1}} du \ ds
\]

Proof: - By definition, we have

\[
L_{A}M(f(t,x)) = F^{A}M(u,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,x) e^{\frac{i(a^{2}-2au+du^{2})}{2} x^{s^{-1}} dt \ dx
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{A}M(u,s) e^{-\frac{i(a^{2}-2au+du^{2})}{2} x^{s^{-1}} du \ ds
\]

\[
\Rightarrow C_{i}(u,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{A}M(u,s) e^{-\frac{i(a^{2}-2au+du^{2})}{2} x^{s^{-1}} du \ ds
\]

(3.1)

where,

\[
C_{i}(u,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ F^{A}M(u,s) e^{-\frac{i(a^{2}-2au+du^{2})}{2} x^{s^{-1}} du \ ds
\]

and

\[
g(t,x) = f(t,x) e^{\frac{i(b^{2}u^{2}-d)}{2} x^{s^{-1}}
\]

From (3.1),

\[
C_{i}(u,s) = FMT(g(t,x)) \left( \frac{w}{b} D_{s} \right)
\]

Where, \( FMT(g(t,x)) \left( \frac{w}{b} D_{s} \right) \) is Fourier-Mellin transform of \( g(t,x) \) with argument \( \frac{u}{b} = w \) and \( s \).

\[
\Rightarrow \frac{du}{b} = dw
\]

\[
\therefore C_{i}(u,s) = FMT(g(t,x)) (w, s)
\]

i.e.,

\[
C_{i}(u,s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t,x) e^{-\frac{i(b^{2}u^{2}-d)}{2} x^{s^{-1}} dt \ dx
\]

http://doi.org/10.29369/ijrbat.2023.010.1.0057
By using inversion formula for Fourier-Mellin transform, we get
\[ g(t,x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_1(u,s) e^{i\omega t} x^{-s} dw ds \]
\[ \Rightarrow g(t,x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{2 \pi b} F^A M(u,s) e^{i\frac{j}{2} \frac{b^2}{s} x^2} e^{i\omega t} x^{-s} dw ds \]
\[ = \frac{1}{2\pi} \sqrt{\frac{1}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{i\frac{j}{2} \frac{b^2}{s} x^2} e^{i\omega t} x^{-s} d\mu d\nu \]
\[ = \frac{1}{2\pi} \sqrt{\frac{1}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{i\frac{j}{2} \frac{b^2}{s} x^2} e^{i\omega t} x^{-s} d\mu d\nu \]
\[ \Rightarrow f(t,x) e^{i\omega t} \]
\[ = \frac{1}{2\pi} \sqrt{\frac{1}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{i\frac{j}{2} \frac{b^2}{s} x^2} e^{i\omega t} x^{-s} d\mu d\nu \]
\[ \Rightarrow f(t,x) e^{i\omega t} \]
\[ = \frac{1}{2\pi} \sqrt{\frac{1}{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{i\frac{j}{2} \frac{b^2}{s} x^2} e^{i\omega t} x^{-s} d\mu d\nu \]

**Parseval’s Theorem for Linear Canonical-Mellin Transform:**

**Theorem:** If \( L_A M \{ f(t,x) \} = F^A M(u,s) \) and \( L_A M \{ g(t,x) \} = G^A M(u,s) \), then
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,x) \overline{g(t,x)} \, dt \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,1-s) \overline{G^A M(u,s)} \, du \, dv \]

**Proof:** We have, by definition
\[ L_A M \{ f(t,x) \} = F^A M(u,s) \]
\[ = \frac{1}{2\pi \sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,x) e^{i(\frac{j}{2} \frac{b^2}{s} x^2)} x^{-s} \, dt \, dx \]

By using inversion formula, we get
\[ f(t,x) \]
\[ = \frac{1}{2\pi \sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{i(\frac{j}{2} \frac{b^2}{s} x^2)} x^{-s} \, du \, dv \]
\[ \Rightarrow f(t,x) \]
\[ = \frac{1}{2\pi \sqrt{2\pi b}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^A M(u,s) e^{i(\frac{j}{2} \frac{b^2}{s} x^2)} x^{-s} \, du \, dv \]

**Conclusion:**

In this paper, we proved inversion formula associated with Linear Canonical-Mellin transform. Subsequently, Parseval’s theorem is proved using inversion formula. The Parseval’s theorem is helpful in signal processing, studying behaviours of random processes and relating functions from one domain to another. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems.

**References:**


Adrian Stern, “Uncertainty principles in linear canonical transform domains and some