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SOME RESULTS ON DIFFERENTIAL POLYNOMIALS

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Preliminaries :

Let f(z) be a transcendental meromorphic function in the finite complex plane. Let P(f) denote the homogeneous differential polynomial in f. As in Hayman [5], such functions will be called differential polynomials in f. Thus a differential polynomial P in f is the sum of a finite number of terms of the form af $^{1_0}\left(f^{(1)}\right)^{l_1}\ldots\left(f^{(m)}\right)^{l_m}$ where $f^{(1)}, f^{(2)}$ ...... are successive derivatives of f and $1_0, 1_1$, ...1_m are non -negative integers. If $1_0 + 1_1 + \ldots\ldots + 1_m = n$ (a fixed positive integer) in every term of P, then P is called homogeneous differential polynomial in f of degree n. in general, if max $(1_0 + 1_1 + \ldots 1_m) = n$ where the maximum is taken over the term of P, then P is said to be a differential polynomial if f of degree at most n.

Definition 1: If f_1, f_2 are meromorphic functions, we denote by $S(r;f_1, f_2)$ a function of r such that

S(r; f_1, f_2) = $o\left(\sum_{i=1}^{2} T(r, f_i\right)$ as $r \to \infty$ through all values if f^i s are of finite orders and outside a set of finite linear measure.

Here we prove the theorems by using following lemmas.

Lemma 1. [9] If P is a homogenous differential polynomial in f of degree

$$n \geq 1, then\, m\left(r, \frac{P}{f^n}\right) = S(r, f).$$

Lemma 2. : [1] Let P be a homogeneous differential polynomial in f of degree n and suppose that P does not involve f. That is, P is a homogeneous differential polynomial of degree n in $f^{(1)}$, $f^{(2)}$,... with coefficients of the form a (z). If P is not a constant and b_1, b_2, ... b_q are distinct elements of c (where q is any positive integer), then

$$n\sum_{i=1}^{q} m(r, b_i, f) + N\left(r, \frac{1}{P}\right) \leq T(r, P) + S(r, f)$$

Lemma 3. [10] Let f_1, f_2 be two non-constant meromorphic functions such that $a_1f_1 + a_2f_2 \equiv 1$, where a_1, a_2 are constants. Then for i = 1, 2

$$T(r, f_i) < \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_i) + S(r, f_1, f_2) \quad \ldots\ldots.(1)$$

Lemma 4 : [1] Let f be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) = S(r, f)$$

and let P be a homogeneous differential polynomial in f. Suppose that P is not a constant. Then the order of P is equal to the order of f and

$$\bar{N}(r, P) + \bar{N}\left(r, \frac{1}{P}\right) = S(r, P)$$

so that $\bar{N}\left(r, \frac{1}{P-a}\right) \neq S(r, P)$ and Θ(a,P) = 0 for all a ≠ 0, ≠ ∞ and there exists no evB for P for distinct zeros in $\bar{C} - \{0, \infty\}$.

A.P. Sing and Dukane [11] have proved the following result.

Theorem A. Let f (z) be a meromorphic function and π_n (f) be a homogenous differential Polynomial of degree n.

$$Let \quad \frac{T(r,\pi_n(f))}{T(r,f)} \to \alpha \ as \ r \to \infty \ \ where \ \alpha \geq n, then$$

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{pmn},$$

where m is the highest derivative occurring in $\pi_n(f)$ and p is the number of terms in $\pi_n(f)$.
Here we shall prove the following improvement of the above theorem. The result here is independent of the number of terms in P.

Theorem 1: Let f (z) be a meromorphic function and P(f) be a homogeneous differential Polynomial of degree n.

$$Let \quad \frac{T(r,P)}{T(r,f)} \to \alpha \ as \ r \to \infty \ \ where \ \alpha \geq n, then$$

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{mn}, \qquad \text{..........(2)}$$

where m is the highest derivative occurring in P(f).

Proof : Let

$$\frac{T(r,P)}{T(r,f)} \to \alpha \ as \ r \to \infty \ \ where \ \alpha \geq n$$

$$Now \quad m(r,P) \leq m\left(r, \frac{P}{f^n}\right) + m(r, f^n)$$

$$= n\ m(r,f) + S(r,f) \qquad by\ \ Lemma\ 1 \qquad \text{....(3)}$$

At a pole of f of order P which is not a pole of any of the coefficient a(z) of P, P has a pole of order at most pn + mn.
So,

$$N(r,P) \leq nN(r,f) + mn\bar{N}(r,f) + S(r,f). \qquad \text{..........(4)}$$

From (3) and (4), we get.

$$T(r,P) \leq nT(r,f) + mn\bar{N}(r,f) + S(r,f).$$

$$Since\ \frac{T(r,P)}{T(r,f)} \to \alpha\ it\ \ fallows\ that$$

$$\alpha T(r,f) \leq nT(r,f) + mn\bar{N}(r,f) + S(r,f). \qquad \text{....(5)}$$

Dividing (5), by T(r,f) and taking limit superior, we get

$$\alpha - n \leq \limsup_{r\to\infty} \frac{nm\bar{N}(r,f)}{T(r,f)} + \limsup_{r\to\infty} \frac{S(r,f)}{T(r,f)}.$$

$$Thus\ \ \alpha - n \leq nm(1 - \Theta(\infty, f))$$

Consequently,

$nm\Theta(\infty,f) \leq nm + n - \alpha$ and so

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{nm}$$

Remark - If m = 1 , n=1 then $\Theta(\infty, f) \leq 2 - \alpha$ this is an interesting generalization of theorem 3 of S.K. Singh and V. N. Kulkarni [12].

Bhoosnurmath [1] has proved the following result.

Theorem B. Let f be a meromorphic function of finite order. If P is a homogeneous differential polynomial in f of degree n and if P does not involve f, then

$$n\sum_{b\in C} \delta(b,f) \leq \delta(O,P) \limsup_{r\to\infty} \frac{T(r,P)}{T(r,f)}$$

and

$$n\sum_{b\in C} \delta(b,f) \leq \Delta(O,P) \liminf_{r\to\infty} \frac{T(r,P)}{T(r,f)}$$

provided that P does not reduce to a constant.
In view of this, we can obtain the following theorem.

Theorem 2. Let f be a meromorphic function of finite order. If P is a homogenous differential polynomial in f of degree n and if P does not involve f, and

$$\liminf_{r\to\infty}\frac{T(r,P)}{T(r,f)}=\limsup_{r\to\infty}\frac{T(r,P)}{T(r,f)}=\alpha, then$$

$$n\sum_{a\in C}\delta(a,f)\le\alpha\delta(O,P) \quad .....(6)$$

$$and \quad n\sum_{a\in C}\delta(a,f)\le\alpha\Delta(O,P) \quad .....(7)$$

provided that P does not reduce to a constant.

Corollary 2.2.1 : Let f be a meromorphic function of finite order with

$$\lim_{r\to\infty}\frac{T(r,P)}{T(r,f)}=\alpha,\ \sum_{a\in C}\delta(a,f)=1$$

If P is a homogeneous differential polynomial in f of degree n and not involving f, then

$$n\le\alpha\delta(O,P)\quad and\quad n\le\alpha\Delta(O,P) \quad ....(8)$$

$$If\ \alpha=n,\ \delta(O,P)=\Delta(O,P)=1 \quad ....(9)$$

provided that P does not reduce to a constant. In particular (10) and (11) hold if f is an entire function

$$\lim_{r\to\infty}\frac{T(r,P)}{T(r,f)}\to\alpha\ and\ \sum_{a\in c}\delta(a,f)=1.$$

Theorem 2.2.3.1 : Let P[f] be a homogeneous differential polynomial such that each term of P involves f, then the order of P [f] and order of f are equal.

Proof. We have

$$P[f]=\sum af^{1_0}\left(f^{(1)}\right)^{1}...\left(f^{(m)}\right)^{1_m}$$

$$T\left(r,P[f]\right)\le T\left(r,a\right)+1_0T\left(r,f\right)+1_1T\left(r,f'\right)+...+1_mT\left(r,f^m\right).$$

$$T\left(r,P[f]\right)\le T\left(r,f\right)\left(l_0+l_1\frac{T\left(r,f'\right)}{T\left(r,f\right)}+1_2\frac{T\left(r,f''\right)}{T\left(r,f\right)}+...+\frac{T\left(r,f^m\right)}{T\left(r,f\right)}\right)+S(r,f) \quad ...(10)$$

we have

$$\limsup_{r\to\infty}\frac{T(r,P)}{T(r,f)}\le n\left[\left(m+1\right)-m\Theta\left(\infty,f\right)\right]$$

$$\limsup_{r\to\infty}\frac{T(r,f^m)}{T(r,f)}\le m+1,\ \ for\ all\ m\ge 1 \quad ....(11)$$

Substituting (11) in (10) , we get

$$\log T(r,P[f])\le\log T(r,f)+\log C$$

$$\rho_p=\limsup_{r\to\infty}\frac{\log T(r,P(f))}{\log r}\le\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\rho_f$$

$$\therefore\ \rho_p\le\rho_f \quad ....(12)$$

Since a zero or a pole of f, which is not a pole of any coefficient a(z) of P, is a pole of $\frac{P}{f^n}$ of degree mn at most, we have

$$N\left(r,\frac{P}{f^n}\right)\le mn\left(\bar N\left(r,\frac{1}{f}\right)+\bar N\left(r,f\right)\right)+S\left(r,f\right)$$

$$nT\left(r,f\right)=T\left(r,f^n\right)$$

$$\le T\left(r,\frac{f^n}{P}\right)+T(r,P)$$

$$\le T\left(r,\frac{P}{f^n}\right)+T(r\,,\,P)+0(1)$$

$$\le T(r,P)+mn\left(\bar{N}\left(r,\frac{1}{f}\right)+\bar{N}(r,f)\right)+S(r,f)$$

$$\le T(r,P)+mn\left(\bar{N}\left(r,\frac{1}{P}\right)+\bar{N}(r,P)\right)$$

Since each term of P involves f term, if follows that

$$\bar{N}(r,f)\le N(r,P)\ and\ \bar{N}\left(r,\frac{1}{f}\right)<\bar{N}\left(r,\frac{1}{P}\right)$$

$$nT(r,f)\le T(r,P)+mn\left(\bar{N}\left(r,\frac{1}{P}\right)+\bar{N}(r,P)\right)+S(r,f)$$

$$\le T(r,P)+mn(2T(r,P))+S(r,f)$$

$$\le (1+2mn)T(r,P)+S(r,f)$$

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}\le\limsup_{r\to\infty}\frac{\log T(r,P[f])}{\log r}=\rho_P \qquad ......(13)$$

From (12) and (13) we conclude that $\rho_P=\rho_f$

Theorem 4 : Let f be a meromorphic function satisfying

$$\bar{N}(r,f)+\bar{N}\left(r,\frac{1}{f}\right)=S(r,f).$$

Let P(f) be a homogeneous differential polynomial which does not reduce to a constant. If G is a mermorphic function such that,

$$\bar{N}\left(r,\frac{1}{G}\right)=S(r,G),$$

then the identity, P + G ≡ 1 is impossible.

This theorem was proved by H.S. Gopalkrishna and S.S. Bhoosnumath [1] but here we improve this theorem removing the condition $\Theta(0,G)>0.$ It is also interesting to note that here we use a different technique.

Proof. Suppose P + G ≡ 1 holds.

Now $\bar{N}$ (r,P) ≤ $\bar{N}$ (r,f) + S(r,f)
= S(r, f), by hypothesis

Therefore $\bar{N}$ (r,P) = S(r, P). ....(14)

Because S(r, f) = S(r,P).
$\bar{N}$ (r, P) = S(r,f) = S(r,P).

Also

$$\bar{N}\left(r,\frac{1}{P}\right)\le(1+mn)\bar{N}\left(r,\frac{1}{f}\right)+mn\bar{N}(r,f)+S(r,f)$$

= S(r,f), by hypothesis

Therefore

$$\bar{N}\left(r,\frac{1}{P}\right)=S(r,f)=S(r,P) \qquad ......(15)$$

Then by Lemma 3 , we have

$$T(r,P) < \bar{N}\left(r,\frac{1}{P}\right) + \bar{N}\left(r,\frac{1}{G}\right) + \bar{N}(r,P) + S(r,P,G)$$

$$T(r,P) < \bar{N}\left(r,\frac{1}{G}\right) + S(r,G) + S(r,P,G), \quad .....(16)$$

by Lemma 4.

Since P + G = 1 holds, is follows that

$T(r,P) \sim T(r,G)$ and $S(r,G) = S(r,P)$. ....(17)

By (16) and (17) , we have

$$T(r,G) < \bar{N}\left(r,\frac{1}{G}\right) + S(r,G) \; Or \; T(r,G) < S(r,G)$$

by hypothesis.

This is a contradiction.

Thus, P + G ≡ 1 is impossible.

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