INTERNATIONAL JOURNAL OF RESEARCHES IN BIOSCIENCES, AGRICULTURE AND TECHNOLOGY
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www.vmsindia.org

TOLERANCE INTERVALS FOR GENERALIZED RAYLEIGH DISTRIBUTION

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Abstract
Aminzadeh (1991) has reported Tolerance Intervals for Rayleigh distribution based on Pivotal Quantity. Recently Kundu and Raqab (2003) have discussed different methods of estimations for the parameters of the Generalized Rayleigh Distribution introduced by Surles and Padgett (2001). In this article, based on Maximum Likelihood Estimators of the parameters, we provide an asymptotic β-expectation and β-content γ-level Tolerance Intervals for this distribution using results due to Shirke et. al. (2004). The performance of the proposed β-expectation Tolerance Interval is assessed though simulations study.

Keywords: β-expectation Tolerance Intervals, β-content γ-level Tolerance Intervals, Expected Coverage, Asymptotic Distribution.

1. Introduction

In general, term Tolerance Interval (TI) is an interval determined from observed values of a random sample for the purpose of drawing inferences about the proportion of a distribution contained in that interval. Usually TI is designed to capture at least a given proportion of some distribution. Two types of TI have received considerable attention in the literature;β-content γ-level TI and β-expectation TI. In order to be more specific about the meaning of TI, let X be a measurable characteristic having a distribution function F(x;θ), $\theta \in \Theta \subseteq \Re$. Let $L(\underline{X})$ and $U(\underline{X})$ be two functions of observations such that $L(\underline{X}) < U(\underline{X})$. Then $(L(\underline{X}), U(\underline{X}))$ is called a β‑content γ‑level TI, if for given β, γ $\in$ (0, 1), $P\{\int_{L(\underline{X})}^{U(\underline{X})} f(t;\theta)dt \geq \beta\} = \gamma$, for every θ ε Θ. If $L(\underline{X})$ and $U(\underline{X})$ are determined so that $E\left[\int_{L(\underline{X})}^{U(\underline{X})} f(t;\theta)dt\right] = \beta$ for every θ∈Θ, then $(L(\underline{X}), U(\underline{X}))$ is called a β-expectation Tolerance Interval, where f(x;θ) is probability density function (pdf) of X. The quantity $\int_{L(\underline{X})}^{U(\underline{X})} f(t;\theta)dt$ is called the sample coverage and $L(\underline{X})$ and $U(\underline{X})$ are called lower and upper tolerance limits, respectively. We set $L(\underline{X}) = -\infty$ to obtain upper TI and set $U(\underline{X}) = \infty$ and obtain lower TI. In the present study, we obtain only upper tolerance limits.

Wilks (1941) treated the problem of determining TIs in pioneer article. Since then a large number of papers dealing with this and other aspects of tolerance limits have appeared in the literature. Jilek(1981) classifies papers according to general results, distribution free results, normal and multivariate normal distributions, gamma, exponential, Weibull and other continuous and discrete distributions. Patel(1986) provided a review, which contains a large collection of known re Aminzadeh (1991) has reported Tolerance Intervals for Rayleigh distribution based on Pivotal Quantity. Recently Kundu and Raqab (2003) have discussed different methods of estimations for the parameters of the Generalized Rayleigh Distribution introduced by Surles and Padgett (2001). sults on β‑content γ‑level TIs for some continuous and discrete univariate distributions.

2. TIs for Exponentiated Scale family of distribution

Shirke, Kumbhar and Kundu (2004) have considered the model

$$\Im = \{F : F_X(x;\theta,\alpha) = [G(x/\theta)]^{\alpha};\ \alpha > 0, \theta > 0\}, \quad (1)$$

where G(·) is known cumulative distribution function (cdf) and named as ***Exponentiated Scale family*** of distribution which is analogous to Lehmann alternatives (see Lehmann (1953)). Under the assumption that all the Cramers regularity conditions are satisfied; by distribution belongs to $\Im$, the TIs are developed for this family.

Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d) with cdf $F_X(x;\theta,\alpha)$; member of $\Im$. The pdf of X_i is

$$f_{X_i}(x;\theta,\alpha)=\begin{cases}\alpha[G(x/\theta)]^{\alpha-1}G'_x(x/\theta) & x\in S,\ \theta,\alpha>0\\ 0 & \text{otherwise,}\end{cases} \tag{2}$$

where S is the support of X, which is independent of parameters and $G'_X(\cdot)$ is the derivative of $G(\cdot)$ with respect to x , and is the pdf of X correxponds to $G(\cdot)$.

2.1 β-expectation Tolerance Interval and its Coverage

Let $\underline{\mu}=(\theta,\alpha)^T$ and $x_\beta(\underline{\mu})\in S$ be the lower β^{th} percentile of $F_X(x;\theta,\alpha)$. Therefore, we have

$$F_X(x_\beta(\underline{\mu});\theta,\alpha)=\beta \qquad \text{for } \beta\in(0,1).$$

This implies

$$X_\beta(\underline{\mu})=\theta G^{-1}(\beta^{1/\alpha}), \tag{3}$$

where $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. Since $\underline{\mu}=(\theta,\alpha)^T$ is unknown, we replace it by its Maximum Likelihood Estimator(MLE). If $\hat{\underline{\mu}}=(\hat\theta,\hat\alpha)^T$ is the MLE of $\underline{\mu}$, then by invariance property of MLE we have MLE of $x_\beta(\underline{\mu})$ is

$$x_\beta(\hat{\underline{\mu}})=\hat\theta G^{-1}(\beta^{1/\hat\alpha}).$$

Therefore, we propose an approximate upper β□-expectation TI as

$$I_1(\underline{X})=(0,\ \hat\theta G^{-1}(\beta^{1/\hat\alpha})). \tag{4}$$

Expected coverage of (4) is given in the following theorem.

Theorem(2.1): Approximate expected coverage of $I_1(\underline{X})$ is given by

$$E[F_X\{X_\beta(\hat{\underline{\mu}});\underline{\mu}\}]=\beta+A(\underline{\mu})\frac{\sigma_1^2}{n}+B(\underline{\mu})\frac{\sigma_2^2}{n}+C(\underline{\mu})\frac{\sigma_{12}}{n},$$

where $A(\underline{\mu})=0.5\alpha(1-\alpha)[G(t)]^{\alpha-2}[G'_\theta(t)]^2+\alpha[G(t)]^{\alpha-1}\left[\frac{G''_{x\theta}(t)G'_\theta(t)}{G'_x(t)}-0.5G''_{\theta\theta}(t)\right]$

$B(\underline{\mu})=\alpha^{-1}[G(t)]^\alpha\log G(t)[1+0.5\alpha\log G(t)]$,

and $C(\underline{\mu})=[G(t)]^\alpha\log G(t)\left[\frac{(\alpha-1)G'_\theta(t)}{G(t)}+\frac{G''_{x\theta}(t)}{G'_x(t)}\right]$

with $t=x/\theta$, $G'_x(t)=\partial G(t)/\partial x$,

$G'_\theta(t)=\partial G(t)/\partial\theta$, $G''_{x\theta}(t)=\partial^2 G(t)/\partial x\partial\theta$,

$G''_{\theta\theta}(t)=\partial^2 G(t)/\partial\theta^2$

while $\frac{\sigma_1^2}{n}$, $\frac{\sigma_2^2}{n}$ are asymptotic variances of $\hat\theta$ and $\hat\alpha$ respectively and

$\frac{\sigma_{12}}{n}=Cov(\hat\theta,\hat\alpha)$ is an asymptotic covariance of $\hat\theta$, $\hat\alpha$.

These quantities can be obtained from **Fisher information matrix I**.

2.2. β-Content γ-level Tolerance Interval

Let $I_2(\underline{X}) = (0,\ \delta \hat{X}_\beta)$ be an upper β-Content γ-level TI for the distribution having cdf (1). The factor $\delta > 0$ is to be determined such that $I_2(\underline{X})$ is a β-Content γ-level TI for $\beta\in(0, 1)$ and $\gamma\in(0, 1)$.

That is
$$P\left[F\left(\delta \hat{X}_\beta;\theta,\alpha\right)\geq\beta\right] = \gamma \cdot$$

Equivalently we get
$$P\left[\hat{X}_\beta \geq \frac{\theta G^{-1}\left(\beta^{1/\alpha}\right)}{\delta}\right] = \gamma \cdot \qquad (5)$$

We note that $\hat{X}_\beta(\underline{\mu}) \rightarrow AN\left(X_\beta(\underline{\mu}),\ \sigma^2(\underline{\mu})/n\right)$,

where $\sigma^2(\underline{\mu}) = H\ I^{-1}H^T$,

I : Fisher information matrix, and $H = \left[\frac{\partial X_\beta(\underline{\mu})}{\partial\theta}, \frac{\partial X_\beta(\underline{\mu})}{\partial\alpha}\right]$.

Using the asymptotic distribution of $\hat{\underline{\mu}}$ we have

$$\delta = \left[1 + \frac{Z_{1-\gamma}\sigma(\underline{\mu})/\sqrt{n}}{\theta G^{-1}(\beta^{1/\alpha})}\right]^{-1}, \qquad (6)$$

where $Z_{1-\gamma}$ is the $100(1-\gamma)^{th}$ percentile of the standard normal variate. As δ depends on both the parameters θ and α replacing θ and α by their respective MLEs, an asymptotic upper β-content γ-level TL of $I_2(\underline{X})$ is proposed as

$$U(\underline{X}) = \left[1 + \frac{Z_{1-\gamma}\sigma(\hat{\underline{\mu}})/\sqrt{n}}{\hat{\theta} G^{-1}(\beta^{1/\hat{\alpha}})}\right]^{-1} X_\beta(\hat{\underline{\mu}}). \qquad (7)$$

3. Tolerance Intervals for Generalized Rayleigh Distribution (GRD)

Surles and Padgett (2001) introduced two-parameter Burr Type X distribution and named as the Generalized Rayleigh Distribution. Kundu and Raqab (2003) discussed different methods of estimations for the parameters of the GRD. Here, we obtain TIs for GRD based on MLE of the parameters. [For expressions of MLE see Kundu and Raqab (2003)]. For α > 0 and θ > 0, the CDF of two-parameter GRD is obtained by substituting $G(x/\theta) = 1 - \exp(-x/\theta)^2$; in (1) as

$$F(x;\alpha,\theta) = \left(1 - \exp\left(-(x/\theta)^2\right)\right)^\alpha \qquad x > 0. \qquad (10)$$

Therefore, GRD has the density function

$$f(x;\alpha,\theta) = 2\frac{\alpha}{\theta^2} x \exp\left(-(x/\theta)^2\right)\left(1 - \exp\left(-(x/\theta)^2\right)\right)^\alpha; \qquad x>0, \qquad (11)$$

where α is the shape parameter and θ is the scale parameter respectively.

3.1 β-expectation TI and its Coverage for GRD

The lower β^{th} percentile of (10) obtained from (3) is

$$X_\beta(\underline{\mu}) = \theta\sqrt{-\ln(1-\beta^{1/\alpha})}.$$

Therefore, an approximate upper β-expectation TI for GRD is obtained from (4) is

$$I_1(\underline{X}) = \left(0,\quad \hat{\theta}\sqrt{-\ln\left(1-\beta^{1/\hat{\alpha}}\right)} \right). \tag{12}$$

Expected coverage of the (12) is given by theorem(2.1),

where $$A(\underline{\mu}) = \alpha(1-f)^{\alpha-2}\left\{(1-\alpha)t^2 f + (1-f)(6t^2-7)\right\}$$

$$B(\underline{\mu}) = \alpha^{-1}(1-f)^{\alpha}\ln(1-f)\{1+0.5\alpha\ln(1-f)\},$$

and $$C(\underline{\mu}) = 2(1-f)^{\alpha-1}\ln(1-f)\left\{(1+\alpha t^2)f + (t^2-1)\right\}\theta^{-1}.$$

with $t = x/\theta$, $f = \exp(-t^2)$.

3.2 . β-Content γ-level TI for GRD

An asymptotic upper limit of β-content γ-level TI for GRD distribution is

$$I_2(\underline{X}) = (\,0,\ \delta\hat{X}_\beta\,),$$

where the constant δ is given by

$$\delta = \left[1 + \frac{Z_{1-\gamma}\sigma(\underline{\mu})/\sqrt{n}}{\theta\sqrt{-\ln(1-\beta^{1/\alpha})}}\right]^{-1}. \tag{13}$$

Therefore, an asymptotic upper β-content γ-level tolerance limit of $I_2(\underline{X})$ is proposed as

$$U(\underline{X}) = \left[1 + \frac{Z_{1-\gamma}\sigma(\hat{\underline{\mu}})/\sqrt{n}}{\hat{\theta}\sqrt{-\ln(1-\beta^{1/\hat{\alpha}})}}\right]^{-1} X_\beta(\hat{\underline{\mu}}). \tag{14}$$

We assess the performance of both types of TI through simulation experiments.

4. Simulation Study

Upper β-expectation TI of GRD:

Note that Theorem-2.1 gives an approximate value of the actual coverage of the interval (12) for GRD. Hence we use simulation to study performance of the proposed β-expectation TI using MLEs of θ and α, namely, $\hat{\theta}$ and $\hat{\alpha}$ respectively. In simulation study

- we generate n (=10, 25, 50,100) observations from the GRD with with θ = 1, and α=2.
- We obtain $\hat{\theta}$ and $\hat{\alpha}$ by solving likelihood equations simultaneously.
- These estimates are then used to compute $X_\beta(\hat{\mu}) = \hat{\theta}\sqrt{-\log(1-\beta^{1/\hat{\alpha}})}$.
- Repeating the process 1000 times we obtain these many estimates of $X_\beta(\underline{\mu})$, and
- Compute expected coverage of the interval $I_1(\underline{X})$.
- Corresponding values are tabulated in the following Table for β = 0.95 and 0.975.

Expected coverage of $I_1(\underline{X})$ TI when θ = 1 and α=2.

β	n			
	10	25	50	100
0.95	0.9162	0.9337	0.9416	0.9458
0.975	0.9500	0.9640	0.9690	09725

Observation: It is observed from Table that for small sample size, n; $I_1(\underline{X})$ underestimates coverage, while for large n; coverage converges to the desired value.

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