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A SOLUTION OF FIRST ORDER LINEAR FUZZY DIFFERENTIAL EQUATION USING FUZZY NUMBER

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ABSTRACT:

The paper talks about solution of linear homogeneous fuzzy differential equation of first order, under fuzzy initial condition. Here symmetrical triangular and trapezoidal fuzzy numbers are used as initial conditions of the said differential equation and using concept of α-cut interval arithmetic a general solution to the problem is proposed.

Keywords : *Fuzzy set, Fuzzy number, Fuzzy differential equation, a-cut of Fuzzy set.*

INTRODUCTION :

Many real time situations are not crisp or precise but they are uncertain in nature, since the variables or parameters included in some mathematical models of engineering, biological and computer systems problems are not precise and this leads to uncertainties of the results, therefore it becomes important to represent such uncertainties in linguistic characterization which can be obtained through fuzzy numbers. First order linear homogeneous differential equations are used in many engineering and science fields for modeling of various problems. For the modeling and handling unclear information in an effective way a fuzzy numbers plays an important role and therefore fuzzy linear differential equation with primary situation with fuzzy number can be used for modeling a subjective concept where boundaries are ambiguous and it can be explored in several fields because of its wide applications.

A term fuzzy differential equations was first time introduce by A. Kandel in year 1978 [1] and in 1982, Dubois-Prade [2] discus the differentiation of fuzzy function in fuzzy environment. In 1985 O. Kaleva [3] discus the distinctive and unique solution of fuzzy differential equation. In 1986, S. Seikkala [4] instigates a solution of initial value problem under fuzzy circumference by extension principle. In 2000, J.Y. Park with H.Han [5] show an occurrence and exclusiveness of fuzzy solution using Hasegawa's functional and successive approximation. In 2002, J. Buckley with T. Feuring [6] gave an universal formulation of fuzzy first order initial value problem and its solution. Y. Chaco-Cano, H. Romain-Flores [7] studied fuzzy differential equations on based on the two different interpretation a new solution for fuzzy differential equation were generated.

In 2010, B.Bede et.al [8] has provided two general solutions exclusively for fuzzy differential equation and their existence. C. Duraisamy with B. Usha [9] solved first order fuzzy differential equation numerically using Runge-Kutta third order method. In 2012, A. Plotnikov with N. Skripnik [10] proved an existence of a fuzzy differential equation through generalized derivative. In 2013, S. Mondal and

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T. Roy [11] have studied the first order fuzzy linear differential equation with generalized fuzzy number as a pioneer condition and provide its solution using Lagrange's Multiplier method.

In the view of above research efforts, here we discussed a solution of Linear Fuzzy Differential Equation of first order using pioneer condition as fuzzy TFN and TrFN number.

PRILIMINARIES

Some important prerequisites.

Definition: **Fuzzy Set**

Fuzzy set 'A' is a collection of objects or elements from the universe of discourse X with membership grade define as $A = \left\{\frac{\mu_A(x)}{x} \middle| x \in X, \mu_A(x) \in [0,1]\right\}$, where $\mu_A(x)$ is membership grade of element x in set A lies between close interval [0,1].

Definition: **Support of Fuzzy Set**

It is a classical set define from fuzzy set A with each element having membership value greater than 0 and it is denoted as $\text{supp}(A) = \{x_k \mid x_k \in X \text{ and } \mu_A(x_k) > 0\}$.

Definition: **Core of Fuzzy Set**

It is a classical set define from fuzzy set A with each element having membership value 1.

And it is denoted as $\text{core }(A) = \{x_k | x_k \in X \text{ and } \mu_A(x_k) = 1\}$.

Definition: α **-cut set**

A classical set define from fuzzy set A with each element having membership value greater or equal for the define value $\alpha \in [0,1]$ and it is represent as $A_\alpha = \{x_k | x_k \in X \text{ and } \mu_A(x_k) \geq \alpha\}$

Definition: **Fuzzy Number**

A fuzzy set 'A' define for every real numbers in R is known as fuzzy number provided must it satisfy the conditions as follows

(i) Fuzzy set is normalized.

i.e. for at least one element $x \in A$, $\mu_A(x) = 1$.

(ii) Support is bounded.

i.e. there are real numbers $\omega_1, \omega_2 \in A$ so that $\forall x \in (-\infty, \omega_1) \,\&\, (\omega_2, \infty)$, $\mu_A(x) = 0$.

(iii) α -cut of fuzzy set A are closed intervals.

i.e. for every $\alpha \in [0,1]$, $A_\alpha = [a_\alpha, \ b_\alpha]$ is closed interval.

Definition: **TFN (Triangular Fuzzy Number)**

A TFN 'A' characterize with three points $[\theta_1 , \theta_2, \theta_3]$ with membership function

$$\mu_A(x) = \begin{cases} 0, & \text{if } x < \theta_1 \text{ and } x > \theta_3 \\ \frac{x-\theta_1}{\theta_2-\theta_1}, & \text{if } \theta_1 \leq x \leq \theta_2 \\ \frac{\theta_3-x}{\theta_3-\theta_2}, & \text{if } \theta_2 \leq x \leq \theta_3 \end{cases}$$

and its α-cut is $A_\alpha = [\theta_1 + \alpha(\theta_2 - \theta_1), \ \theta_3 - \alpha(\theta_3 - \theta_2)] = [\theta_1 + \alpha\mathcal{L}, \ \theta_3 - \alpha\mathcal{R}]$,

where $\mathcal{L} = (\theta_2 - \theta_1) > 0$ and $\mathcal{R} = (\theta_3 - \theta_2) > 0$.

If $\mathcal{L} = \mathcal{R}$ then triangular fuzzy number $A = [\theta_1 , \theta_2, \theta_3]$ is known as symmetric TFN.

Definition: **TrFN (Trapezoidal Fuzzy Number)**

A trapezoidal fuzzy number 'A' characterize by four point $[\theta_1 , \theta_2, \theta_3, \theta_4]$ with membership function

$$\mu_A(x) = \begin{cases} 0, & \text{if } x<\theta_1 \text{ and } \& \ x> \theta_4 \\ \frac{x-\theta_1}{\theta_2-\theta_1}, & \text{if } \theta_1 \leq x \leq \theta_2 \\ 1, & \text{if } \theta_2 \leq x \leq \theta_3 \\ \frac{\theta_4-x}{\theta_4-\theta_3}, & \text{if } \theta_3 \leq x \leq \theta_4 \end{cases}$$

and its α-cut is $A_\alpha = [\theta_1 + \alpha(\theta_2 - \theta_1), \ \theta_4 - \alpha(\theta_4 - \theta_3)] = [\theta_1 + \alpha\mathcal{P}, \ \theta_3 - \alpha\mathcal{Q}]$,

where $\mathcal{P} = (\theta_2 - \theta_1) > 0$ and $\mathcal{Q} = (\theta_4 - \theta_3) > 0$.

If $\mathcal{P} = \mathcal{Q}$ then triangular fuzzy number $A = [\theta_1 , \theta_2, \theta_3, \theta_4]$ is known as symmetric TrFN.

Definition: **Interval Arithmetic's**

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are any two closed intervals define on real line $\mathbb{R}$ such that $a_1 \leq a_2$ and $b_1 \leq b_2$ then it addition and subtraction are obtained as follows

i) Addition: $A + B = [a_1, a_2] + [b_1, b_2] = [(a_1 + b_1) \ , \ (a_2 + b_2)]$

ii) Subtraction: $A - B = [a_1, a_2] - [b_1, b_2] = [(a_1 - b_2) \ , \ (a_2 - b_1)]$.

Definition: **First Order Linear Homogeneous Fuzzy Differential Equation (FOLHFDE):**

An equation of the type $\frac{dy}{dt} + ky = 0$, $y(t_0) = y_0$ is called as first order homogeneous fuzzy differential equation if starting value y_0 is fuzzy number and hence the solution is also fuzzy.

Definition: **Strong and Weak Solution**

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Let the first order homogeneous fuzzy differential equation is $\frac{dy}{dt}+ky=0$, $y(t_0)=y_0$, where k is any constant and y_0 is fuzzy number. The solution $y(t)$ be a strong solution if it's α-cut, $y(t,\alpha)=[y_1(t,\alpha),y_2(t,\alpha)]$ satisfy following conditions (a) $\frac{\partial y_1(t,\alpha)}{\partial\alpha}>0$ and (b) $\frac{\partial y_2(t,\alpha)}{\partial\alpha}<0$, otherwise it is a weak solution.

Case Study

1. Solution of FOLHFDE using Triangular Fuzzy number:

Consider a fuzzy differential equation

$$\frac{dy}{dt}+ky=0 \qquad \text{-----------------(1)}$$

with given condition $\tilde{y}(t_0)=\widetilde{y_0}=[\theta_1\ ,\ \theta_2,\ \theta_3]$ as TFN.

Let $\tilde{y}(t)$ is the solution of equation (1) with α-cut, $y(t,\alpha)=[y_1(t,\alpha),y_2(t,\alpha)]$

And α-cut of given TFN is $y(t_0,\alpha)=[y_1(t_0,\alpha),y_2(t_0,\alpha)]=[\theta_1+\alpha\mathcal{L},\ \theta_3-\alpha\mathcal{R}]$, $\forall\alpha\in[0,1]$

Case 1: If $k>0$ then from α-cut of (1) we got

$$\frac{dy_1(t,\alpha)}{dt}+ky_1(t,\alpha)=0 \quad \text{and} \quad \frac{dy_2(t,\alpha)}{dt}+ky_2(t,\alpha)=0$$

And their solutions are $y_1(t,\alpha)=(\theta_1+\alpha\mathcal{L})e^{-k(t-t_0)}$ --------------------(2)

$$y_2(t,\alpha)=(\theta_3-\alpha\mathcal{R})e^{-k(t-t_0)} \qquad \text{--------------------(3)}$$

Here $\frac{\partial y_1(t,\alpha)}{\partial\alpha}=\mathcal{L}\ e^{-k(t-t_0)}>0$ and $\frac{\partial y_2(t,\alpha)}{\partial\alpha}=-\mathcal{R}e^{-k(t-t_0)}<0$

Now, core of solution, that is for $\alpha=1$ we obtain, $y_1(t,1)=\theta_2\ e^{-k(t-t_0)}=y_2(t,\alpha)$ and support of solution, that is for $\alpha=0$ we get $y_1(t,0)=\theta_1\ e^{-k(t-t_0)}$ and $y_2(t,0)=\theta_3\ e^{-k(t-t_0)}$

Thus the required strong solution is $\tilde{y}(t)=[\theta_1\ ,\ \theta_2,\ \theta_3]e^{-k(t-t_0)}=\widetilde{y_0}\ e^{-k(t-t_0)}$

Case 2: If $k<0$ i.e $k=-m$, where m > 0 then from α-cut of (1) we got

$$\frac{dy_1(t,\alpha)}{dt}-my_2(t,\alpha)=0 \quad \text{and} \quad \frac{dy_2(t,\alpha)}{dt}-my_1(t,\alpha)=0$$

And their solutions are

$$y_1(t,\alpha)=\frac{1}{2}\{(\theta_1+\theta_3)+\alpha(\mathcal{L}-\mathcal{R})\}e^{m(t-t_0)}+\frac{1}{2}\{(\theta_1-\theta_3)+\alpha(\mathcal{L}+\mathcal{R})\}e^{-m(t-t_0)}$$

$$y_2(t,\alpha)=\frac{1}{2}\{(\theta_1+\theta_3)+\alpha(\mathcal{L}-\mathcal{R})\}e^{m(t-t_0)}-\frac{1}{2}\{(\theta_1-\theta_3)+\alpha(\mathcal{L}+\mathcal{R})\}e^{-m(t-t_0)}$$

Since for symmetric TFN , $\mathcal{L}=\mathcal{R}$ therefore we got

$$y_1(t,\alpha)=\frac{1}{2}(\theta_1+\theta_3)e^{m(t-t_0)}+\frac{1}{2}\{(\theta_1-\theta_3)+\alpha(\mathcal{L}+\mathcal{R})\}e^{-m(t-t_0)} \qquad \text{--------------(4)}$$

$$y_2(t,\alpha)=\frac{1}{2}(\theta_1+\theta_3)e^{m(t-t_0)}+\frac{1}{2}\{(\theta_3-\theta_1)-\alpha(\mathcal{L}+\mathcal{R})\}e^{-m(t-t_0)} \qquad \text{--------------(5)}$$

Here $\frac{\partial y_1(t,\alpha)}{\partial\alpha}=\frac{1}{2}(\mathcal{L}+\mathcal{R})\ e^{-m(t-t_0)}>0$ and $\frac{\partial y_2(t,\alpha)}{\partial\alpha}=-\frac{1}{2}(\mathcal{L}+\mathcal{R})\ e^{-m(t-t_0)}<0$

Now, core of solution, that is for $\alpha=1$, $y_1(t,1)=y_2(t,1)=\frac{1}{2}(\theta_1+\theta_3)e^{m(t-t_0)}$ and

support of solution, that is for $\alpha=0$, $y_1(t,0)=\frac{1}{2}(\theta_1+\theta_3)e^{m(t-t_0)}+\frac{1}{2}(\theta_1-\theta_3)e^{-m(t-t_0)}$

$$y_2(t,0)=\frac{1}{2}(\theta_1+\theta_3)e^{m(t-t_0)}+\frac{1}{2}(\theta_3-\theta_1)e^{-m(t-t_0)}$$

Thus the required strong solution is $\tilde{y}(t)=\frac{1}{2}(\theta_1+\theta_3)e^{m(t-t_0)}+\frac{1}{2}\ \tilde{O}\ e^{-m(t-t_0)}$

Where $\tilde{O}=[\theta_1-\theta_3\ ,\ 0\ ,\ \theta_3-\theta_1\ ]$ is TFN.

2. Solution of FOLHFDE using Trapezoidal Fuzzy Number:

Consider fuzzy differential equation

$$\frac{dy}{dt}+ky=0 \qquad \text{-----------------(6)}$$

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with given condition $\tilde{y}(t_0) = \tilde{y_0} = [\theta_1 , \theta_2, \theta_3, \theta_4]$as TrFN.

Let $\tilde{y}(t)$ is the solution of equation (1) with α-cut, $y(t,\alpha) = [y_1(t,\alpha) , y_2(t,\alpha)]$

And α-cut of given TFN is $y(t_0,\alpha) = [y_1(t_0,\alpha) , y_2(t_0,\alpha)] = [\theta_1 + \alpha\mathcal{P},\ \theta_4 - \alpha\mathcal{Q}],\ \forall\alpha \in [0,1]$

Case 1: If $k > 0$ then from α-cut of (1) we got

$\frac{dy_1(t,\alpha)}{dt} + ky_1(t,\alpha) = 0$ and $\frac{dy_2(t,\alpha)}{dt} + ky_2(t,\alpha) = 0$

And their solutions are $y_1(t,\alpha) = (\theta_1 + \alpha\mathcal{P})e^{-k(t-t_0)}$ --------------------(7)

$y_2(t,\alpha) = (\theta_4 - \alpha\mathcal{Q})e^{-k(t-t_0)}$ --------------------(8)

Here $\frac{\partial y_1(t,\alpha)}{\partial\alpha} = \mathcal{P}\, e^{-k(t-t_0)} > 0$ and $\frac{\partial y_2(t,\alpha)}{\partial\alpha} = -\mathcal{Q}e^{-k(t-t_0)} < 0$

Now, core of solution, i.e. for $\alpha = 1$ we get

$\tilde{y}(t,1) = [y_1(t,1), y_2(t,1)] = [\,\theta_2\, e^{-k(t-t_0)},\ \theta_3\, e^{-k(t-t_0)}]$

and support of solution, i.e. for $\alpha = 0$ we get

$\tilde{y}(t,0) = [y_1(t,0), y_2(t,0)] = [\,\theta_1\, e^{-k(t-t_0)},\ \theta_4\, e^{-k(t-t_0)}]$

Thus the required strong solution is $\tilde{y}(t) = [\theta_1 , \theta_2, \theta_3, \theta_4\,]e^{-k(t-t_0)} = \tilde{y_0}\ e^{-k(t-t_0)}$

Case 2: If $k < 0$ i.e $k = -m$, where m > 0 then from α-cut of (1) we got

$\frac{dy_1(t,\alpha)}{dt} - my_2(t,\alpha) = 0$ and $\frac{dy_2(t,\alpha)}{dt} - my_1(t,\alpha) = 0$

And their solutions are

$$y_1(t,\alpha) = \frac{1}{2}\{(\theta_1 + \theta_4) + \alpha(\mathcal{P} - \mathcal{Q})\}e^{m(t-t_0)} + \frac{1}{2}\{(\theta_1 - \theta_4) + \alpha(\mathcal{P} + \mathcal{Q})\}e^{-m(t-t_0)}$$

$$y_2(t,\alpha) = \frac{1}{2}\{(\theta_1 + \theta_4) + \alpha(\mathcal{P} - \mathcal{Q})\}e^{m(t-t_0)} - \frac{1}{2}\{(\theta_1 - \theta_4) + \alpha(\mathcal{P} + \mathcal{Q})\}e^{-m(t-t_0)}$$

Since for symmetric TrFN , $\mathcal{P} = \mathcal{Q}$ therefore we got

$$y_1(t,\alpha) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}\{(\theta_1 - \theta_4) + \alpha(\mathcal{P} + \mathcal{Q})\}e^{-m(t-t_0)} \qquad \text{--------------(9)}$$

$$y_2(t,\alpha) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}\{(\theta_4 - \theta_1) - \alpha(\mathcal{P} + \mathcal{Q})\}e^{-m(t-t_0)} \qquad \text{--------------(10)}$$

Here $\frac{\partial y_1(t,\alpha)}{\partial\alpha} = \frac{1}{2}(\mathcal{P} + \mathcal{Q})\, e^{-m(t-t_0)} > 0$ and $\frac{\partial y_2(t,\alpha)}{\partial\alpha} = -\frac{1}{2}(\mathcal{P} + \mathcal{Q})\, e^{-m(t-t_0)} < 0$

Now, core of solution, i.e. for $\alpha = 1$ we obtain

$$y_1(t,1) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}(\theta_2 - \theta_3)e^{-m(t-t_0)}$$

And $y_2(t,1) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}(\theta_3 - \theta_2)e^{-m(t-t_0)}$

and support of solution, i.e. for $\alpha = 0$ we obtain

$$y_1(t,0) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}(\theta_1 - \theta_4)e^{-m(t-t_0)}$$

$$y_2(t,0) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}(\theta_4 - \theta_1)e^{-m(t-t_0)}$$

Thus the required strong solution is $\tilde{y}(t) = \frac{1}{2}(\theta_1 + \theta_4)e^{m(t-t_0)} + \frac{1}{2}\,\tilde{\Psi}\, e^{-m(t-t_0)}$

Where $\tilde{\Psi} = [\theta_1 - \theta_4\,,\ \theta_2 - \theta_3\ ,\ \theta_3 - \theta_2\,,\ \theta_4 - \theta_1\ ]$ is TrFN.

CONCLUSION :

In our research we proposed a strong general solution of Homogeneous First Order Linear Fuzzy Differential Equation where starting condition as a symmetric TFN and symmetric TrFN. To solve the fuzzy differential model here we used a method of α-cut interval arithmetic and for the further work we suggest to solve Non-Homogeneous Linear Fuzzy Differential Equation under the initial condition as fuzzy number by same method.

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