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EXTENDED WAVELET TRANSFORM AND ITS APPLICATION IN SIGNAL PROCESSING

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ABSTRACT:
Multi resolution techniques are deeply related to image/signal processing, biological and computer vision, scientific computing, optical data analysis. Improving quality of noisy signals/images has been an active area of research in many years. Wavelet transform can achieve good scarcity for spatially localized details, such as edges and singularities. Wavelet transform is emerged as the most effective technique for signal processing and image analysis as an alternative to Fourier analysis especially when the signals are random, comprised of fluctuations of different scales and where the very short and very long waves are present in the same signal.
In the present work a strong relationship between Wavelet transform and fractional Fourier transform is presented. Analytical behavior of extended Wavelet transform is exploited which can be viewed as extension of fractional Fourier transform. This interpretation of Wavelet transform in terms of fractional Fourier transform is then used in the far reaching applications especially in the field of signal processing and in particular, in the field of long range optical fiber transmission; which has been an active area of research ever since the introduction of multi resolution techniques in the fractal representation of modulated signals. Experimental results have shown that the wavelet based models have better performance over the other transform techniques ever applied for signal processing.

Keywords:- Fractional Fourier transform, Wavelet transform, Signal processing, Image Denoising.

INTRODUCTION :

The wavelet transform has been shown to be a successful tool for dealing with transient signals, data compression, bandwidth reduction, time dependent frequency analysis of short transient signals, optical correlators, sound analysis, representation of the human retina and representation of fractal aggregates. The different wavelet components are scaled and shifted versions of mother wavelet. In many applications, especially in the time–frequency analysis of a signal, the standard Fourier transform is not adequate because the Fourier transform of the signal does not contain any local information. There have been several investigation into additive noise suppression in signal and images using wavelet transform[3]

Fractional Fourier transform:

Fractional Fourier transform are closely related to a particular wavelet transform family [1,10].

The one dimensional fractional FT with parameter α of $f(x)$ denoted by $R^{\alpha}f(x)$ performs a linear operation, given by the integral transform

$$[R^{\alpha}f(x)(p) = F_{\alpha}(\xi) = \int_{-\infty}^{\infty} K_{\alpha}(x,p)f(x)dx,$$

where the kernel $K_{\alpha}(x,p) = \frac{e^{i\alpha/2}}{\sqrt{2\pi i sin\alpha}} exp\left[\frac{i}{2sin\alpha}((x^2+p^2)cos\alpha - 2xp)\right]$

$$= C_{1\alpha} \exp\left(C_{2\alpha}\left((x^2+p^2)cos\alpha - 2xp\right)\right),$$

where $C_{1\alpha} = (2\pi i sin\alpha)^{-1/2} \exp\left(i\alpha/2\right)$ and $C_{2\alpha} = \frac{1}{2}sin\alpha$

The Classical Wavelet Transform

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The classical wavelet transform of a function 'f' with respect to a given admissible mother wavelet 'g' is defined on wavelet domain coefficients at scale a and translation b [6]

$$(W_g f)(a,b) = |a|^{-1/2} \int f(x)\, g^*((x-b)/a)dx \qquad .. (1.1)$$

Relationship between Wavelet transform and fractional Fourier transform (FrFT)

$$(Wf)\ (a,b) = \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{c(\propto)|tan^{1/2}\propto|^n} \quad [R^{\propto}f]\left(\frac{b}{sec\propto}\right)$$

where $b = psec\propto,\ a = tan^{1/2}\propto,\ C(\propto) = \frac{\exp\left(\frac{i\propto}{2}\right)}{(2\pi i\, sin\propto)^{1/2}}$

$$= \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto(2\pi i\, sin\propto)^{\frac{1}{2}}}{\exp\left(\frac{i\propto}{2}\right)\left|tan^{\frac{1}{2}}\propto\right|^n} \int_{-\infty}^{\infty} f(x) \quad K_{\propto}(x,p)\, dx$$

$$= \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto(2\pi i\, sin\propto)^{\frac{1}{2}}}{\exp\left(\frac{i\propto}{2}\right)\left|tan^{\frac{1}{2}}\propto\right|^n} \int_{-\infty}^{\infty} f(x)\ \frac{exp\left(\frac{i\propto}{2}\right)}{\sqrt{2\pi i \sin\alpha}} exp\left[\frac{i}{2\sin\propto}(x^2+p^2)\cos\propto -2xp\right] dx$$

$$= \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{\left|tan^{\frac{1}{2}}\propto\right|^n} \int_{-\infty}^{\infty} f(x)\ exp\left[\frac{i}{2\sin\propto}(x^2+p^2)\cos\propto -2xp\right]$$

$$(Wf)(a,b) = \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{\left|tan^{\frac{1}{2}}\propto\right|^n} \quad (\text{FrFT})$$

Generalization of Extended Wavelet Transform

The testing function space $E(R^n)$

An infinitely differentiable complex valued function $\emptyset$ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_{a,}$, where, $S_{a,} = \{x : x \in R^n, |x| \le a, a > 0\}$,

$$\gamma_{E_m}(\emptyset) = \sup_{x\in I}|D_x^m \emptyset(x)| < \infty \qquad ... (3.1)$$

Thus, $E(R^n)$ will denote the space of all $\emptyset \in E(R^n)$ with support contained in S_a. Moreover, we say that f is a Wavelet transformable, if it is a member of E^*, the dual space of E.

Definition of generalized Extended Wavelet transform

The distributional generalized Wavelet transform of $f(x,y) \in E^*(R^n)$ is defined by,

$$WT\{f(x)\}(p) = W_\alpha(a,b) = \langle f(x),\ K_\alpha(x,p)\rangle, \qquad ...(3.2)$$

where $K_\alpha(x,p) = \frac{e^{\frac{i}{2}}b^2 sin^2\alpha}{\left|tan^{\frac{1}{2}}\alpha\right|^n} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]}$, $b = psec\propto,\ \ a = tan^{\frac{1}{2}}\propto,$

$$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+p^2)\cos\alpha - 2xp]}. \qquad ...(3.3)$$

where $C_{1\alpha} = \frac{e^{i/2}b^2 sin^2\alpha}{|tan^{1/2}\alpha|^n},\ C_{2\alpha} = \frac{1}{2\sin\alpha}$

where right hand side of equation (3.2) is meaningful because according to the theorem 3.2, $K_\alpha(x,p) \in E$ and $f \in E^*$.

Analytical Behaviour of generalized Extended Wavelet Transform

In this section we have shown that generalized Wavelet transform is analytic in terms of definition.

Analyticity theorem

Let $f(x) \in E^*(R^n)$ and its generalized Wavelet transform be defined by, $WT\{f(x)\}(p) = W_\alpha(a,b) = \langle f(x),\ K_\alpha(x,p)\rangle$, then $W_\alpha(a,b)$ is analytic on C^n, if $suppf \subset S_q$, where $S_q = \{x : x\epsilon R^n, |x| \le q,\ q > 0\}$. Moreover, $W_\alpha(a,b)$is differentiable and $D_p^m\{W_\alpha(a,b)\} = \langle f(x),\ D_p^m\{K_\alpha(x,p)\}\rangle$.

...(4.1)

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Proof- Let $p:(p_1,p_2,---,p_j,---p_n)\epsilon C^n$, we first prove that

$\frac{\partial}{\partial p_j}\{W_\alpha(p)\}=\langle f(x),\frac{\partial}{\partial p_j}\{K_\alpha(x,p)\}\rangle,$

For fixed $p_j\neq 0$, choose two concentric circles C and C' with centre at p_j with radii r and r_1 respectively such that $0<r<r_1<|p_j|$.

let Δp_j be a complex increment satisfying $0<|p_j|<r$.

Consider,

$$\frac{W_\alpha(p_j+\Delta p_j)-W_\alpha(p_j)}{\Delta p_j}-\langle f(x),\frac{\partial}{\partial p_j}\{K_\alpha(x,p)\}\rangle=\langle f(x),\psi_{\Delta p_j}(x)\rangle \quad ...(4.2)$$

$$\Rightarrow \frac{1}{\Delta p_j}\{\langle f(x),K_\alpha(x,p_j+\Delta p_j)\rangle-\langle f(x,y),K_\alpha(x,p_j)\rangle\}-\langle f(x),\frac{\partial}{\partial p_j}\{K_\alpha(x,p)\}\rangle=\langle f(x),\psi_{\Delta p_j}(x)\rangle$$

$$\Rightarrow\langle f(x),\frac{1}{\Delta p_j}\{K_\alpha(x,p_j+\Delta p_j)-K_\alpha(x,p_j)\}-\frac{\partial}{\partial p_j}\{K_\alpha(x,p)\}\rangle=\langle f(x),\psi_{\Delta p_j}(x)\rangle,$$

where $\psi_{\Delta p_j}(x)=\frac{1}{\Delta p_j}\{K_\alpha(x,p_j+\Delta p_j)-K_\alpha(x,p_j)\}-\frac{\partial}{\partial p_j}\{K_\alpha(x,p)\}$.

For any fixed $x,y\epsilon R^n$ and fixed integer $m=(m_1,m_2,----m_n)$,

$D_x^m\{K_\alpha(x,p)\}$

$$=D_x^m\left\{\frac{e^{i/2}b^2 sin^2\alpha}{|tan^{1/2}\alpha|^n}e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}\right\}$$

$$=\frac{e^{i/2}b^2sin^2\alpha}{|tan^{1/2}\alpha|^n}\ e^{\frac{i}{2}p^2cot\alpha}\ D_x^m e^{\frac{i}{2sin\alpha}[x^2cos\alpha-2xp]}$$

$$=C_{1\alpha}\sum_{h=0}^{m}C_mC_\alpha\,(xcos\alpha-p)^{m-2h}e^{(m-h)t}\ ,$$

where $C_m=\frac{m!}{(m-2h)!h!}(i)^{m-h}(2)^{m-2h},\quad C_\alpha=(C_{2\alpha})^{m-h}cos^h\alpha,$

$$t=ic_{2\alpha}[(x^2+p^2)\cos\alpha-2xp],\ C_{1\alpha}=\frac{e^{i/2}b^2sin^2\alpha}{|tan^{1/2}\alpha|^n}$$

Since for any fixed $x\in R^n$, fixed m and $0<\alpha\leq\frac{\pi}{2}$, $D_x^m\{K_\alpha(x,p)\}$ is analytic inside and on C',

we have by Cauchy's integral formula,

$$D_x^m\psi_{\Delta p_j}(x)=\frac{1}{2\pi i}D_x^m\int_{C'}K_\alpha(x,\tilde{p},)\left[\frac{1}{\Delta p_j}\left(\frac{1}{z-p_j-\Delta p_j}-\frac{1}{z-p_j}\right)-\frac{1}{(z-p_j)^2}\right]dz,$$

where $\tilde{p}=p_1,p_2,---,p_{j-1},z,p_{j+1},---p_n$

$$=\frac{\Delta p_j}{2\pi i}\int_{C'}\left[\frac{A(x,\tilde{p})}{(z-p_j-\Delta p_j)(z-p_j)^2}\right]dz$$

But for all $z\epsilon C'$ and x is restricted to a compact subset of R^n, $0<\alpha\leq\frac{\pi}{2}$,

$A(x,\tilde{p})=D_x^mK_\alpha(x,\tilde{p})$ is bounded by a constant Q.

Moreover, $|z-p_j-\Delta p_j|>r_1-r>0\&|z-p_j|=r_1$.

Therefore we have,

$$\left|D_x^m\psi_{\Delta p_j}(x)\right|=\left|\frac{\Delta p_j}{2\pi i}\int_{C'}\frac{A(x,\tilde{p})}{(z-p_j-\Delta p_j)(z-p_j)^2}dz\right|$$

$$\leq\frac{|\Delta p_j|}{2\pi}\int_{C'}\frac{Q}{(r_1-r){r_1}^2}|dz|$$

$$\leq\frac{|\Delta p_j|Q}{(r_1-r)r_1}$$

Thus as $|\Delta p_j|\to 0$,$D_x^m\psi_{\Delta p_j}(x,y)$ tends to zero uniformly on the compact subset of R^n, therefore it follows that $\psi_{\Delta p_j}(x)$ converges in $E(R^n)$ to zero. Since $f(x)\in E^*$, we conclude that equation (3.4.2) also tends to zero.

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Therefore, $F_\alpha(p)$ is differentiable with respect to p_j. But this is true for all $j = 1,2,---n$, Hence $W_\alpha(a,b)$ is analytic on C^n and $D_p^m\{F_\alpha(p)\} = \langle f(x), D_p^m\{K_\alpha(x,p)\}\rangle$.

Inversion formula for extended Wavelet transform

The Generalized Wavelet transform is given by

$$(Wf)\ (a,b) = \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{\left|tan^{\frac{1}{2}}\propto\right|^n}\int_{-\infty}^{\infty} f(x)\ exp\left[\frac{i}{2\sin\propto}(x^2+p^2)\cos\propto -2xp\right] d\propto$$

then by inversion it is possible to recover $f(x)$ by means of the inversion formula

$$f(x) = \int_{-\infty}^{\infty} W_\alpha(a,b)\,\widetilde{K}_\alpha(x,p)dp,$$

where $\widetilde{K}_\alpha(x,p) = \frac{cosec\,\alpha}{2\pi}\left(\frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{\left|tan^{\frac{1}{2}}\propto\right|^n}\right)^{-1} e^{\frac{-i}{2}[(x^2+p^2)cot\alpha - 2(xp)cosec\alpha]}$

Proof - $WT\{f(x)\}(p) = W_\alpha(a,b)$=

$$= \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{\left|tan^{\frac{1}{2}}\propto\right|^n}\int_{-\infty}^{\infty} f(x)\ exp\left[\frac{i}{2\sin\propto}(x^2+p^2)\cos\propto -2xp\right],\ b = psec\propto,\ \ a = tan^{1/2}\propto$$

$$= C_{1\alpha}\int_{-\infty}^{\infty} f(x)e^{iC_{2\alpha}[(x^2+p^2)cos\alpha - 2(xp)]}dxdy$$

$$\text{where } C_{1\alpha} = \frac{exp\left(\frac{i}{2}\right)b^2 sin^2\propto}{\left|tan^{\frac{1}{2}}\propto\right|^n}, C_{2\alpha} = \frac{1}{2sin\alpha}$$

$$= e^{iC_{2\alpha}[(p^2)cos\alpha]}\int_{-\infty}^{\infty} C_{1\alpha}f\ (x)\ e^{iC_{2\alpha}[(x^2)cos\alpha - 2(xp)]}dx$$

$$e^{-iC_{2\alpha}[(p^2)cos\alpha]}\ W_\alpha(a,b) = \int_{-\infty}^{\infty} C_{1\alpha}f\ (x)\ e^{iC_{2\alpha}[(x^2)cos\alpha - 2(xp)]}dx$$

$$= C_{1\alpha}\int_{-\infty}^{\infty} f\ (x)\ e^{\frac{i}{2}(x^2)cot\alpha}\,e^{-i(xp)cosec\alpha}\,dx$$

$$= \int_{-\infty}^{\infty} g\ (x)\ e^{-i(pcosec\alpha)x}\,dx$$

where, $g(x) = C_{1\alpha}f(x)e^{\frac{i}{2}(x^2)cot\alpha}$ ... (4.3)

$$e^{-iC_{2\alpha}[(p^2)cos\alpha]}\ WT\{f(x)\}(p) = \int_{-\infty}^{\infty} g\ (x)\ e^{-iAx}\,dx,$$

where $A = pcosec\alpha$,

$= F[g(x)](A)$ ... (4.4)

$$e^{-iC_{2\alpha}[(p^2)cos\alpha]}\ W_\alpha(a,b)\left(\frac{A}{cosec\alpha}\right) = F[g(x)](A) = G(A) \quad ...(4.5)$$

The right hand side is the Fourier transform of $g(x)$ with argument A. Invoking the Fourier inversion we can write

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G\ (A)e^{i(Ax)}dA \quad ...(4.6)$$

Now putting the value of $g(x)$ by using (4.3)

$$C_{1\alpha}f(x)e^{\frac{i}{2}(x^2)cot\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G\ (A)e^{i(Ax)}dA$$

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$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iC_{2\alpha}[(p^2)cos\alpha]} W_\alpha(a,b) \left(\frac{A}{cosec\alpha}\right)$$

$$e^{i(xp)cosec\alpha} cosec\alpha \, dp$$

$$f(x) = \int_{-\infty}^{\infty} W_\alpha(a,b) \widetilde{K}_\alpha(x,p) dp, \quad where$$

$$\widetilde{K}_\alpha(x,p) = \frac{cosec\,\alpha}{2\pi} \left(\frac{exp\left(\frac{i}{2}\right) b^2 sin^2 \propto}{\left|tan^{\frac{1}{2}} \propto\right|^n} \right)^{-1} e^{\frac{-i}{2}[(x^2+p^2)cot\alpha - 2(xp)cosec\alpha]} \qquad \ldots (4.7)$$

Image denoising by extended Wavelet transform

Noise arises due to imperfect instruments used in signal processing, problems with the data acquisition process, and interference which can degrade the data of interest. Also, noise can be introduced due to compression and transmission errors [11]. De-noising or the noise reduction is an essentially required process to enhance the estimation process of signal/image reconstruction of the captured signal.

The basic idea behind wavelet denoising, or wavelet thresholding is that the wavelet transform concentrates signal and image features in a few large-magnitude wavelet coefficients. Wavelet coefficients which are small in value are typically noise and you can "shrink" those coefficients or remove them without affecting the signal or image quality. After thresholding the coefficients, you can reconstruct the data using the inverse wavelet transform [4].

Wavelet denoising is a simple operation, which aims at reducing noise in a noisy image. It is performed by selecting the wavelet coefficients below a certain thresholdand setting them to zero as follows:

$$y_\lambda = y_\lambda, y_\lambda \geq t_\lambda$$
$$= 0, y_\lambda < t_\lambda$$

where t_λ is the threshold and λ is the index.

The threshold used is $t_\lambda = k\sigma_\lambda \sigma$, for some scale k, where σ is an estimation of the standard deviation of the noise, $\sigma = \frac{Median}{0.6745}$, and σ_λ is an approximation value for the standard deviation of each wavelet coefficient estimated by using the Monte-Carlo simulation .

To conduct the performance factor analysis, the peak signal to noise ratio (PSNR) and root mean square error (RMSE) measures are used.

$$PSNR = 10\log_{10}\left(\frac{\sum_{i=1}^{N} x^2(i)}{\sum_{i=1}^{N}[x(i)-\hat{x}(i)]^2}\right), RMSE = \frac{1}{N}\sqrt{\sum_{i=1}^{N}[x(i)-\hat{x}(i)]^2}$$

where $x(i)$ is the original source signal, $\hat{x}(i)$ is the separated signal, i is the sample index and N is the number of samples of the signal.

The test is: Higher the value of PSNR with minimum value of RMSE, better the performance of the denoising model.

For example, simulations performed on noisy mixed 256X256 'CT' image on Matlab® R 7.9 on a core i7 2.2 GHz PC and wavelet transform via USFFT software package.

The denoising results (PSNR in dB) in both the experiments are presented below:

Image	Noisy Image	Wavelet thresholding	RMSE
CT (Random noise)	21.19 (m=0, σ^2=0.01)	23.89	0.13443

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CT (Gaussian noise)	19.77(m=0, σ^2=0.01)	27.70	0.12569

Form the experimental results, it is seen that the wavelet based denoising operation gives increased PNSR with minimum RMSE for sample image. Similar simulations can be performed for optical data, satellite images, seismic data.

CONCLUSION :

In the present work Extension of wavelet transform via fractional Fourier transform is obtained. Analytical behavior of extended Wavelet transform is presented. The extended version of wavelet transform has far-reaching applications especially in the field of signal processing and in particular, in the field of long range optical fibre transmission. Experimental results have shown that the wavelet based models have better performance over the other transform techniques ever applied for signal processing.

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