Bianchi Type VI, Magnetized Cosmological Model in $f(R, T)$ Theory of Gravitation

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Abstract: In the $f(R)$ theory of gravity, we have studied electromagnetic fields in Bianchi type-VIo space time by considering the general case. It is found that if the study is confined to the case of $f(R) = \mu R$, it is observed that the convergent and isotropic solution of the metric function can be evolved with the components of the vector potentials.

Keyword: Bianchi Type VIo, Electromagnetic Field, $f(R)$ theory of gravity, isotropy, constant vector potential.

1. Introduction

Basically, two kinds of alternative accelerated expansion of the universe have been proposed for this unexpected observational phenomenon. One is negative pressure known as dark energy (DE) which induces a late-time accelerating cosmic expansion. The other is the modified gravity, which originate from the idea that the general relativity is incorrect in the cosmic scale and therefore need to be modified.

In order to explain the nature of the DE and accelerated expansion, a variety of theoretical models have been proposed in literature. There are several modified gravity theories like $f(R)$ gravity formulated by Nojiri and Odinstov [4,6]. The idea of introducing additional terms of the Ricci scalar to the Einstein-Hilbert action did not begin years ago with the $f(R)$ theory of gravity paper by Carroll et.al [5]. He explained the presence of a late time cosmic acceleration of the universe in $f(R)$ theory of gravity.

The $f(T)$ theory of gravitation formulated by Nojiri and Odinstov [3]. To justify the current expansion of the universe come from modified or alternative theories of gravity, $f(T)$ Theory of gravity is one such example which has been recently developed. This theory is a generalized version of teleparallel gravity in which Weitzenbok connection is used instead of Levi-Civita connection. The interesting feature of this theory is that it may explain the current acceleration without involving dark energy. In our opinion, one of interesting and prospective version of modified gravity theories is the $f(R, T)$ gravity proposed by Harko et al [8,9] and Myrzakulov [10]. In $f(R, T)$ theory of gravity, cosmic acceleration may result not only due to geometrical contribution to the total cosmic energy density but it is also depends on matter contents. Many authors have investigated different problems within the scope of $f(R, T)$ theory. Bijan Saha [11] has studied the interacting scalar and electromagnetic fields in Bianchi type I universe.

Solanke and Karade [12] have studied electromagnetic field in $f(R, T)$ gravity in Bianchi type-III spacetime. Our interest is to explore the role of scalar and electromagnetic field played in the amended $f(R, T)$ gravity in other Bianchi types or other metric universe. In this paper we consider axially symmetric metric universe.

2. Gravitational field equations of $f(R, T)$ gravity

The field equation of $f(R, T)$, theory due to Harko[8-9] are deduced by varying the action.

\[ s = \int f(R, T) \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x, \]

where $L_m$ are Lagrangian and other symbols have their usual meaning.

Varying the action (1.1) with respect to $g^{\mu\nu}$, which yields as
\[ \delta S = \frac{1}{2x} \left[ f_R(R,T) \frac{\partial R}{\partial \theta} + f_T(R,T) \frac{\partial T}{\partial \theta} + \frac{\delta f(R,T)}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial \theta} + \frac{2\sqrt{-g}}{\partial \theta} \right] \sqrt{-g} d^4x \]

Constraining \( \delta S = 0 \) from equation (2.2) upon integration, we can obtain

\[ f_R(R,T) g^{ij} - \frac{1}{2} f(R,T) g_{ij} + \left( g_{ij} \nabla^i \nabla_j - \nabla_i \nabla_j \right) f_R(R,T) = \delta T_{ij} - f_T(R,T) \left( \theta_j + \theta_i \right) \]

where \( \nabla_i \) is the constant derivative.

Replaced \( f(R,T) \) by \( f(R) \) in (2.3), we obtain

\[ f_R(R) g^{ij} - \frac{1}{2} f(R) g_{ij} + \left( g^{ij} g_{ij} \nabla^i \nabla_j - g^{ij} g_{ij} \nabla_j \right) f_R(R) = \delta T_{ij} \]

Taking trace of equation (2.4), we get

\[ f_R(R) g^{ij} - \frac{1}{2} f(R) g_{ij} + \left( g^{ij} g_{ij} \nabla^i \nabla_j - g^{ij} g_{ij} \nabla_j \right) f_R(R) = \delta T_{ij} \]

We consider the particular case \( f(R) = \mu R \).

It follows notation

\[ f(R) = \frac{\partial F(R)}{\partial R} = \frac{\partial}{\partial R} \mu R = \mu \]

The field (2.4) with the aid of (2.6), reduces to

\[ \mu R g^{ij} - \frac{1}{2} \mu R g_{ij} + \left( g^{ij} g_{ij} \nabla^i \nabla_j - g^{ij} g_{ij} \nabla_j \right) \mu = \delta T_{ij} \]

The equation (2.5) with the aid of (2.6), reduces to

\[ \nabla^i \nabla_j \mu = -\frac{1}{3} xT + 2 \left( \frac{1}{3} \mu R - \frac{1}{3} \mu R \right) \\
\frac{1}{3} xT + \frac{1}{3} \mu R = 0 \]

3. Energy momentum tensor for electromagnetic field

Energy momentum tensor for electromagnetic field is given by

\[ \frac{1}{2x} \left[ f_T(R,T) \frac{\partial T}{\partial \theta} + f_T(R,T) \frac{\partial T}{\partial \theta} + \frac{\delta f(R,T)}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial \theta} + \frac{2\sqrt{-g}}{\partial \theta} \right] \sqrt{-g} d^4x \]
\[ T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta\sqrt{-g} L_m}{\delta g^{ij}} = -\frac{2}{\sqrt{-g}} \left( -\frac{\sqrt{-g} M_m}{\delta g_{ij}} + L_m \frac{\delta\sqrt{-g}}{\delta g_{ij}} \right) = 2 \frac{\delta M_m}{\delta g_{ij}} + 2 \frac{\delta\sqrt{-g}}{\delta g_{ij}} L_m, \]

\[ = 2 \frac{\delta M_m}{\delta g_{ij}} - \frac{\sqrt{-g}}{g_{mm}} L_m \frac{\delta g_{mm}}{\delta g_{ij}} = 2 \frac{\delta M_m}{\delta g_{ij}} - g_{mm} \frac{\delta g_{mm}}{\delta g_{ij}} = g_{mn} \frac{\delta g_{mn}}{\delta g_{ij}}, \]

\[ T_{ij} = 2 \frac{\delta M_m}{\delta g_{ij}} - g_{ij} L_m - 2 \frac{\delta M_m}{\delta g_{ij}} - L_m g_{ij}, \] (3.1)

where \( F_{kl} \) is the electromagnetic field.

\[ \frac{\partial M_m}{\partial g_{ij}} = \frac{\partial}{\partial g_{ij}} F_{kl} \]

\[ = \frac{1}{4} \frac{\partial}{\partial g_{ij}} F_{kl} \]

\[ = \frac{1}{2} g_{ik} \frac{\partial}{\partial g_{k}} \]

(3.2)

Using (3.3), the equation (3.1) reduces to

\[ T_{ij} = \frac{1}{4} F_{kl} F^{kl} - F_{kl} F_{ij}. \] (3.4)

4. The metric and field equations.

We consider the axially symmetric metric in the form

\[ ds^2 = dt^2 - a_t^2 dx^2 - a^2 dx^2_e^{-2m^2} dy^2 - a_3^2 e^{2m^2} dz^2, \] (4.1)

where \( a_t, a_2, \) and \( a_3 \) are functions of time \( t \).

5. Electromagnetic field tensor \( F_{ij} \)

The electromagnetic field tensor is given by

\[ F_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}, \] (5.1)

To achieve the capability with non-static space time (4.1), we assume electromagnetic vector potential in the form

\[ A_t = [a(\alpha) v_1(t), v_2(t), v_3(t), v_4(t)]. \] (5.2)

From (5.1) and (5.2) we can easily deduce

\[ F_{14} = u v_1, \quad F_{24} = v_2, \quad F_{34} = v_3, \] (5.3a)

\[ F_{41} = \frac{u v_1}{a_1^2} F_{32} = -\frac{v_2}{a_2^2 e^{-2m^2}} F_{33} = -\frac{v_3}{a_3^2 e^{2m^2}}, \] (5.3b)

\[ F_4^2 = -\frac{v_2}{a_2^2 e^{-2m^2}}, \quad F_3^3 = -\frac{v_3}{a_3^2 e^{2m^2}}, \] (5.3c)

\[ F_1^4 = -u v_1, \quad F_2^4 = -v_2, \quad F_3^4 = -v_2. \] (5.3d)

From above equations we can compute...
Using (3.4), we establish the following nonzero components of the energy momentum tensor of material field

\[ T_1^i = \frac{1}{2} \left[ \frac{u^2 v_1^2}{a_1} - \frac{v_2^2}{a_2} e^{-2m^2x} - \frac{v_1^2}{a_3} e^{-2m^3x} \right] \tag{5.4a} \]

\[ T_2^i = \frac{1}{2} \left[ \frac{u^2 v_1^2}{a_1} - \frac{v_2^2}{a_2} e^{-2m^2x} - \frac{v_1^2}{a_3} e^{-2m^3x} \right] \tag{5.4b} \]

\[ T_3^i = \frac{1}{a_3^2} e^{-2m^3x} \left[ \frac{a_2 a_3 v_1^2}{2a_1} e^{2m^2x} + \frac{a_2^2 v_2^2}{2a_1} e^{2m^2x} - \frac{v_1^2}{2} \right] \tag{5.4c} \]

\[ T_4^i = \frac{1}{2} \left[ \frac{u^2 v_1^2}{a_1} + \frac{v_2^2}{a_2} e^{-2m^2x} + \frac{v_1^2}{a_3} e^{-2m^3x} \right] \tag{5.4d} \]

From equations (5.4a) (5.4b) (5.4c) and (5.4d), we can deduced the components of energy Tensor as follows

\[ T_i^i = 0 \tag{5.5} \]

6. Maxwell’s Equation

The Maxwell Equations are given by

\[ \frac{(v_1)'}{v_1} + \frac{v_1^2}{v_1^2} \left[ \frac{a_2 + a_3 - a_1}{a_2 + a_3 - a_1} \right] = 0 \tag{6.1a} \]

\[ \frac{(v_2)'}{v_2} + \frac{v_2^2}{v_2^2} \left[ \frac{a_1 + a_3 - a_2}{a_1 + a_3 - a_2} \right] = 0 \tag{6.1b} \]

\[ \frac{(v_3)'}{v_3} + \frac{v_3^2}{v_3^2} \left[ \frac{a_1 + a_2 - a_3}{a_1 + a_2 - a_3} \right] = 0 \tag{6.1c} \]

\[ u = c \tag{6.1d} \]

where \( c \) is a constant.

7. Solution of vector potential

Considering the trivial solution component of Ricci tensor form field (2.9), we get
\[
\frac{V_1 V_2}{v_1 v_2} = 0, \quad (7.1a)
\]

\[
\frac{V_1 V_3}{v_1 v_3} = 0, \quad (7.1b)
\]

\[
\frac{V_2 V_3}{v_2 v_3} = 0, \quad (7.1c)
\]

From (6.1a to 6.1c), we write

\[
\frac{V_1 V_2}{v_1 v_2} = \frac{V_1 V_3}{v_1 v_3} = \frac{V_2 V_3}{v_2 v_3} = 0, \quad (7.2)
\]

which further imply

\[
\frac{V_1}{v_1} = \frac{V_2}{v_2} = \frac{V_3}{v_3} = \frac{D}{D}, \quad (7.3)
\]

where \( D \) is some unknown function of \( t \).

Using (7.3), we obtain

\[
v_1 = K_1D, \quad v_2 = K_2D, \quad v_3 = K_3D, \quad (7.4)
\]

where \( k's \) are constants of integration.

8. Solution of field equations.

As in Solanke and Karade (12), we consider

\[
\frac{u^2 v_1^2}{a_1^2} + \frac{v_1^2}{a_2^2 e^{-2m^2x}} + \frac{v_2^2}{a_3^2 e^{2m^2x}} = \frac{u^2 v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2 e^{-2m^2x}} + \frac{v_3^2}{a_3^2 e^{2m^2x}} \left( \frac{D}{D} \right)^2 = - \left( \frac{D}{D} \right)^2 \]

Now our plan is to express the components of \( T^i_j \) in terms of \( T^4_4 \)

\[
T_1^1 = \frac{u^2 v_1^2}{2a_1^2} - \frac{v_1^2}{2a_2^2 e^{-2m^2x}} - \frac{v_2^2}{2a_3^2 e^{2m^2x}} = - T_4^4 - \left( \frac{D}{D} \right)^2, \quad (8.1a)
\]

\[
T_2^2 = \frac{u^2 v_2^2}{2a_1^2} + \frac{v_2^2}{2a_2^2 e^{-2m^2x}} - \frac{v_3^2}{2a_3^2 e^{2m^2x}} = T_4^4 + \frac{v_2^2}{a_3^2 e^{-2m^2x}} \left( \frac{D}{D} \right)^2, \quad (8.1b)
\]

\[
T_3^3 = \frac{u^2 v_3^2}{2a_1^2} - \frac{v_3^2}{2a_2^2 e^{-2m^2x}} + \frac{v_1^2}{2a_3^2 e^{2m^2x}} = - T_4^4 + \frac{v_3^2}{a_3^2 e^{-2m^2x}} \left( \frac{D}{D} \right)^2, \quad (8.1c)
\]

\[
T_4^4 = \frac{u^2 v_4^2}{2a_1^2} + \frac{v_4^2}{2a_2^2 e^{-2m^2x}} + \frac{v_3^2}{2a_3^2 e^{2m^2x}} = - \frac{1}{2} \left( \frac{D}{D} \right)^2, \quad (8.1d)
\]

By using (7.4), we get trace of energy momentum tensor as
With the help of (7.3), we can write the equation (6.1) as

\[ T = I \left( \frac{D}{D} \right)^2 - \frac{1}{2} I \left( \frac{D}{D} \right)^2 = 0, \tag{8.2} \]

Considering the non-trivial components of Ricci tensor from field equation (2.9)

\[ \mu R^i_j + \frac{x}{2} x T g^j_i = x T^j_i, \tag{8.3} \]

\[
\mu \left[ \begin{array}{c}
\frac{2m^4}{a^2} + \frac{a_1}{a_1} + \frac{a_2 a_3}{a_2 a_3} + \frac{a_1 a_3}{a_1 a_3} \\
\frac{a_2}{a_2} + \frac{a_1 a_2}{a_1 a_2} + \frac{a_2 a_3}{a_2 a_3} \\
\frac{a_3}{a_3} + \frac{a_1 a_3}{a_1 a_3} + \frac{a_2 a_3}{a_2 a_3}
\end{array} \right] = x \left[ -T^4_4 + \frac{v^2_1}{a^2_2} e^{-2m x} \left( \frac{D}{D} \right)^2 \right],
\tag{8.3a}
\]

\[
\mu \left[ \begin{array}{c}
\frac{a_2}{a_1} + \frac{a_1 a_2}{a_1 a_2} + \frac{a_2 a_3}{a_2 a_3} \\
\frac{a_2}{a_2} + \frac{a_1 a_3}{a_1 a_3} + \frac{a_2 a_3}{a_2 a_3} \\
\frac{a_3}{a_3} + \frac{a_1 a_3}{a_1 a_3} + \frac{a_2 a_3}{a_2 a_3}
\end{array} \right] = x \left[ -T^4_4 + \frac{v^2_1}{a^2_3} e^{-2m x} \left( \frac{D}{D} \right)^2 \right], \tag{8.3b}
\]

By using (7.4), above equations leads to

\[-\frac{2m^4}{a^2} + \frac{a_1}{a_1} + \frac{a_2 a_3}{a_2 a_3} + \frac{a_1 a_3}{a_1 a_3} = 0, \tag{8.4a}\]

\[
\frac{a_2}{a_2} + \frac{a_1 a_2}{a_1 a_2} + \frac{a_2 a_3}{a_2 a_3} = 0, \tag{8.4b}\]

\[
\frac{a_3}{a_3} + \frac{a_1 a_3}{a_1 a_3} + \frac{a_2 a_3}{a_2 a_3} = 0. \tag{8.4c}\]

With the help of (7.3), we can write the equation (6.1) as

\[
\left( \frac{D}{D} \right)^2 + \frac{D}{D} \left[ \frac{a_2}{a_2} + \frac{a_3}{a_3} - \frac{a_1}{a_1} \right] = 0, \tag{8.4d}\]

\[
\left( \frac{D}{D} \right)^2 + \frac{D}{D} \left[ \frac{a_1}{a_1} + \frac{a_3}{a_3} - \frac{a_2}{a_2} \right] = 0 \tag{8.4e}
\]

\[
\left( \frac{D}{D} \right)^2 + \frac{D}{D} \left[ \frac{a_1}{a_1} + \frac{a_2}{a_2} - \frac{a_3}{a_3} \right] = 0. \tag{8.4f}\]
This equation further imply that
\[ \frac{a_1}{a_1} = \frac{a_2}{a_2} = \frac{a_3}{a_3}, \]  
which on Integration with respect to \( t \), we get
\[ a_1 = K_4 a_2, \quad a_2 = K_5 a_3, \quad a_3 = K_6 a_1, \]  
where \( k \)'s are constants of integration.

From (8.4) and (8.5), we get
\[ \frac{a_2}{a_2} + 2 \left( \frac{a_2}{a_2} \right)^2 = 0, \]  
\[ \frac{a_3}{a_3} + 2 \left( \frac{a_3}{a_3} \right)^2 = 0, \]  
Upon integration, we get
\[ a_2 = (3k_1 t + 3k_8)^\frac{1}{3}, \]  
\[ a_3 = (3k_2 t + 3k_10)^\frac{1}{3}, \]  
With the help of (8.6) and (8.8a), we have
\[ a_1 = (3k_1 t + 3k_12)^\frac{1}{3}. \]  
By using the equations (8.5), (8.8a), (8.8b) and (8.8c) we get
\[ \frac{D}{D} = \frac{d}{t} \frac{D}{D} d = 0, \quad D = C, \]  
where \( C \) is constant.

From equation (8.9) and (8.4a) we get
\[ m = 0. \]  
Using equations (8.9), the line element (4.1) reduces to
\[ ds^2 = dt^2 - (3d_1 t + 3kd_2)^\frac{1}{3} \left[ dx^2 + e^{-2w(x,y)} dy^2 + e^{2w(x,y)} dz^2 \right]. \]  

**Conclusion**

In this paper, we have considered the particular case \( f(R) = \mu R \) model in Bianchi type VIo.

1) It is observed that convergent non-singular, isotropic solution can be evolved for the metric function and the components of vector potential.

2) Model shows that universe expand algebraically in \( f(R) = \mu R \) theory of gravity.

3) The metric function in non-static space time admits constant value at early time of the universe. \( T \) tends to zero and after that the metric function start increasing with increase in cosmic time and finally diverge to infinity as time tend to infinity .This shows that the universe expand and approaches to infinite volume.

4) It is also interesting to note that the investigated model is free from singularity. Hence the model approaches isotropic for the anytime.
References


