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New Operators for Two Dimensional Fractional Fourier Transform

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Abstract:
An integral transform is a particular kind of mathematical operator. The Fourier transform is invariant in modulus to translations in frequency, but not to dilations. Fourier transform has many applications such as watermarking, pattern recognition, in signal processing etc.
In this paper we have introduced some new operators of two dimensional fractional Fourier transform. These operators are applicable while solving heat equation, wave equation, Schrodinger equation, etc.
Keywords: Testing function space, Fourier transform, fractional Fourier transform, Two dimensional fractional Fourier transform

Introduction

Fractional Fourier transform may be considered as fractional power of the classic Fourier transform. The first idea of fractional power of the Fourier operator appears in 1929, it has been introduced by Wiener. Then concept of the fractional Fourier transform was described by Condon and was later introduced for signal processing in 1980 by Namias as a Fourier transform of fractional order. Sumiyoshi et al also made an interesting generalization on fractional Fourier transform in 1994. The fractional Fourier transform is generalization of classical Fourier transform. The canonical fractional Fourier transform was introduced more than 60 years ago in the mathematical literature [1], after that, it was reinvented for applications in quantum mechanics [2] [7], optics [3] [4] [5], and signal processing [4]. Fractional Fourier transform is applied to the Ku-band ground-based radar imaging of ground moving train [6]. Fractional Fourier Transform is used to Designing A Re-Configurable Architecture by Using Systolic Array method [7]. Broadband interference excision in spread spectrum communication system by using fractional Fourier transform[17].

Recently the fractional Fourier transform has been reintroduced twice with optical applications in mind. Hence Fourier transform and fractional Fourier transform has important properties and applications as describe in [10,11,12,13,14,15,16].

In our previous work we proved an inversion theorem, convolution theorem of two dimensional fractional Fourier transform [8, 9]. In this paper we have proved some important operators are applicable while solving heat equation, wave equation, Schrodinger equation, etc.

Definitions

The testing function space $E(R^n)$

An infinitely differentiable complex valued function $\emptyset$ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_{a,b}$, where

$$S_{a,b} = \{x, y: x, y \in R^n, |x| \le a, |y| \le b, a > 0, b > 0\},$$

$$\gamma_{E_{m,n}}(\emptyset) = \sup_{x,y \in I} \left| D_{x,y}^{m,n} \emptyset(x,y) \right| < \infty \qquad (2.2.1)$$

Thus, $E(R^n)$ will denote the space of all $\emptyset \in E(R^n)$ with support contained in $S_{a,b}$. Moreover, we say that f is a two dimensional fractional Fourier transformable, if it is member of E^*, the dual space of E.

Definition of generalized two dimensional fractional Fourier transform (FRFT)

The distributional two dimensional fractional Fourier transform of $f(x,y) \in E^*(R^n)$ is defined by ,

$$FRFT\{f(x,y)\} = F_\alpha(p,q) = \langle f(x,y), K_\alpha(x,y,p,q) \rangle, \quad \text{where}$$

$$K_\alpha(x,y,p,q) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}} \, e^{\frac{1}{2\sin\alpha}[(x^2+y^2+p^2+q^2)\cos\alpha - 2(xp+yq)]},$$

where right hand side of equation (2.2.2) is meaningful because according to the theorem 2.2.2 $K_\alpha(x,y,p,q) \in E$ and $f \in E^*$.

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Result:

$$\frac{\partial^2}{\partial p \partial q}\{FRFT[f(x,y)]\}(p,q) = -\{FRFT[(pcot\alpha - xcosec\alpha)$$
$$(qcot\alpha - ycosec\alpha)]f(x,y)\}(p,q)$$

Proof- By definition of (2DFRFT) two dimensional fractional Fourier transform

$$\frac{\partial^2}{\partial p \partial q}\{FRFT[f(x,y)]\}(p,q)$$

$$= \frac{\partial^2}{\partial p \partial q}\sqrt{\frac{1 - icot\alpha}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \ \ e^{\frac{i}{2}[(x^2+y^2)cot\alpha + (p^2+q^2)cot\alpha] - i(xp+yq)cosec\alpha} dxdy$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \frac{\partial}{\partial p}\left[e^{\frac{i}{2}x^2 cot\alpha + \frac{i}{2}p^2 cot\alpha - ixpcosec\alpha}\right] \ \frac{\partial}{\partial q}\left[e^{\frac{i}{2}y^2 cot\alpha + \frac{i}{2}q^2 cot\alpha - iyqcosec\alpha}\right]$$
$$dxdy$$

$$= -\{FRFT[(pcot\alpha - xcosec\alpha) \ \ (qcot\alpha - ycosec\alpha)]f(x,y)\}(p,q)$$

Result:

$$\frac{\partial^4}{\partial p^2 \partial q^2}\{FRFT[f(x,y)]\}(p,q) \ = -A\{FRFT[f(x,y)]\} + \{cot^2\alpha cosec^2\alpha \ \ [(xq+py)^2 + 2i$$
$$(xp+yq)sin\alpha + 2xypq]FRFT[f(x,y)] - cosec\alpha cot\alpha$$
$$[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2\alpha +$$
$$i(x^2+y^2)cosec\alpha] \quad FRFT[f(x,y)] + x^2y^2cosec^4\alpha FRFT[f(x,y)]$$

Proof- By using (3.1)

$$\frac{\partial^2}{\partial p \partial q}\{FRFT[f(x,y)]\}(p,q) = -\{FRFT[(pcot\alpha - xcosec\alpha) \ \ (qcot\alpha - ycosec\alpha)]$$
$$f(x,y)\}(p,q)$$

Again Differentiate with respect to p and q partially

$$\frac{\partial^4}{\partial p^2 \partial q^2}\{FRFT[f(x,y)]\}(p,q) \ = -\frac{\partial^2}{\partial p \partial q}\{FRFT[(pcot\alpha - xcosec\alpha) \ \ (qcot\alpha - ycosec\alpha)]$$
$$f(x,y)\}(p,q)$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \frac{\partial^2}{\partial p \partial q}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \ (pcot\alpha - xcosec\alpha) \ \ (qcot\alpha - ycosec\alpha)$$
$$\left[e^{\frac{i}{2}(x^2+y^2+p^2+q^2)cot\alpha - i(xp+yq)cosec\alpha}\right] \ \ dxdy$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \ \int_{-\infty}^{\infty} \frac{\partial}{\partial p} \ (pcot\alpha - xcosec\alpha) \ \left[e^{\frac{i}{2}x^2 cot\alpha + \frac{i}{2}p^2 cot\alpha - ixpcosec\alpha}\right]$$
$$\int_{-\infty}^{\infty} \frac{\partial}{\partial q}(qcot\alpha - ycosec\alpha) \ \left[e^{\frac{i}{2}y^2 cot\alpha + \frac{i}{2}q^2 cot\alpha - iyqcosec\alpha}\right] \ f(x,y)dxdy$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[-p^2q^2cot^2\alpha + 2p^2yqcot^3\alpha cosec\alpha \ - p^2y^2cot^2\alpha cosec^2\alpha$$
$$+ip^2cot^3\alpha + 2q^2xpcot^3\alpha cosec\alpha \ - xypqcot^2\alpha cosec^2\alpha + 2xpy^2cot\alpha cosec^3\alpha$$
$$-2ixpcot^2\alpha cosec\alpha - x^2q^2 \ cot^2\alpha cosec^2\alpha - 2x^2yqcot\alpha cosec^3\alpha \ - x^2y^2cosec^4\alpha$$
$$+ix^2cot\alpha cosec^2\alpha \ + iq^2\alpha cot^3\alpha - 2iyqcosec\alpha cot^2\alpha + iy^2cosec^2\alpha cot\alpha + cot^2\alpha]$$
$$f(x,y) \ \ e^{\frac{i}{2}(x^2+y^2+p^2+q^2)cot\alpha - i(xp+yq)cosec\alpha} dxdy$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{A - cot^2\alpha cosec^2\alpha \ \ [x^2q^2 + p^2y^2 + 2i(xp+yq)sin\alpha + 4xypq]$$
$$+[2pq(py-qx)cot^2\alpha + 2xy(py+xq) \ \ cosec^2\alpha + i(x^2+y^2)cosec\alpha] \ \ cosec\alpha cot\alpha$$

$-x^2y^2cosec^4\alpha\}\quad f(x,y)\; e^{\frac{i}{2}(x^2+y^2+p^2+q^2)cot\alpha - i(xp+yq)cosec\alpha}dxdy$

where $A = [1 - p^2q^2cot^2\alpha + i(p^2+q^2)cot\alpha]cot^2\alpha$

$-A + cot^2\alpha cosec^2\alpha FRFT\; [x^2q^2 + p^2y^2 + 2i(xp+yq)sin\alpha + 4xypq]\{f(x,y)\}$

$-cosec\alpha cot\alpha FRFT\; +[2pq(py - qx)cot^2\alpha + 2xy(py+xq)\; cosec^2\alpha$

$+i(x^2+y^2)cosec\alpha]\{f(x,y)\} + cosec^4\alpha FRFT\{x^2y^2f(x,y)\}$

$= -A\{FRFT[f(x,y)]\} + \{cot^2\alpha cosec^2\alpha\; [(xq+py)^2 + 2i\,(xp+yq)sin\alpha +$
$2xypq]FRFT[f(x,y)] - cosec\alpha cot\alpha\; [2pq(py+xq)cot^2\alpha + 2xy(py+xq)$
$cosec^2\alpha + i(x^2+y^2)cosec\alpha]\quad FRFT[f(x,y)] + x^2y^2cosec^4\alpha FRFT[f(x,y)]$

Result: $\{FRFT[xyf(x,y)]\}(p,q) \quad = -\left(pqcos^2\alpha + sin^2\alpha\frac{\partial^2}{\partial p\partial q}\right)$
$\{FRFT[f(x,y)]\} + cos\alpha FRFT\{(yp+xq)f(x,y)\}$

Proof- By using result (3.1)

$$\frac{\partial^2}{\partial p\partial q}\{FRFT[f(x,y)]\}(p,q) = -FRFT\{[(pcot\alpha - xcosec\alpha)\quad(qcot\alpha - ycosec\alpha)]\; f(x,y)\}(p,q)$$

$$= -FRFT\{[pqcot^2\alpha - pycosec\alpha cot\alpha - xqcosec\alpha cot\alpha + xycosec^2\alpha]\; f(x,y)\}(p,q)$$

$$= -\sqrt{\frac{1-icot\alpha}{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[pqcot^2\alpha - cosec\alpha cot\alpha\;(yp+xq) + xycosec^2\alpha]$$
$$f(x,y)\; e^{\frac{i}{2}(x^2+y^2+p^2+q^2)cot\alpha - i(xp+yq)cosec\alpha}dxdy$$

$$FRFT[xyf(x,y)] = -pqcos^2\alpha FRFT\{f(x,y)\} + cos\alpha\; FRFT[(yp+xq)f(x,y)$$
$$-sin^2\alpha\frac{\partial^2}{\partial p\partial q}\; FRFT\{f(x,y)\}$$
$$= -(pqcos^2\alpha + sin^2\alpha\frac{\partial^2}{\partial p\partial q})\; FRFT\{f(x,y)\} + cos\alpha\; FRFT[(yp+xq)f(x,y)$$

Result: $\{FRFT[x^2y^2f(x,y)]\}(p,q)$

$$= -cos^2\alpha sin^2\alpha[1 - p^2q^2cot^2\alpha + i(p^2+q^2)cot\alpha]\{FRFT[f(x,y)]\}$$
$$-cot^2\alpha[(xq+py)^2 + 2y(xp+yq)sin\alpha + 2xypq]\;\{FRFT[f(x,y)]\}$$
$$+cos\alpha sin^2\alpha[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2\alpha$$
$$+i(x^2+y^2)cosec\alpha]\;\{FRFT[f(x,y)]\} + sin^4\alpha\frac{\partial^4}{\partial p^2\partial q^2}\{FRFT[f(x,y)]\}$$
$$(p,q)$$

Proof- By using result (3.2)

$$\frac{\partial^4}{\partial p^2\partial q^2}\{FRFT[f(x,y)]\}(p,q) = -A\{FRFT[f(x,y)]\} + \{cot^2\alpha cosec^2\alpha\;[(xq+py)^2 + 2i$$
$$(xp+yq)sin\alpha + 2xypq]FRFT[f(x,y)] - cosec\alpha cot\alpha$$
$$[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2\alpha +$$
$$i(x^2+y^2)cosec\alpha]\quad FRFT[f(x,y)] + x^2y^2cosec^4\alpha FRFT[f(x,y)]$$

where $A = [1 - p^2q^2cot^2\alpha + i(p^2+q^2)cot\alpha]cot^2\alpha$

$$-x^2y^2cosec^4\alpha FRFT[f(x,y)] = -A\{FRFT[f(x,y)]\} + \{cot^2\alpha cosec^2\alpha\;[(xq+py)^2$$
$$+2i(xp+yq)sin\alpha + 2xypq]\; FRFT[f(x,y)] - cosec\alpha cot\alpha$$
$$[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2\alpha + i(x^2+y^2)cosec\alpha]$$

$$FRFT[f(x,y)] - \frac{\partial^4}{\partial p^2 \partial q^2}\{FRFT[f(x,y)]\}(p,q)$$
$$-x^2y^2FRFT[f(x,y)] = -cos^2\alpha sin^2\alpha\ [1 - p^2q^2cot^2\alpha + i(p^2+q^2)cot\alpha]$$
$$\{FRFT[f(x,y)]\} - cos^2\alpha[(xq+py)^2\ +2i(xp+yq)sin\alpha$$
$$+2xypq]\ FRFT[f(x,y)] + cos\alpha sin^2\alpha$$
$$[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2\alpha + i(x^2+y^2)cosec\alpha]$$
$$FRFT[f(x,y)] - sin^4\alpha \frac{\partial^4}{\partial p^2 \partial q^2}\{FRFT[f(x,y)]\}(p,q)$$

New operators of two dimensional fractional Fourier transform tabular form

S.N.	Operator	Result
1	$\frac{\partial^2}{\partial p \partial q}\{FRFT[f(x,y)]\}(p,$	$-\{FRFT[(pcot\alpha - xcosec\alpha)$ $(qcot\alpha - ycosec\alpha)]f(x,y)\}(p,q)$
2	$\frac{\partial^4}{\partial p^2 \partial q^2}\{FRFT[f(x,y)]\}(p$	$-A\{FRFT[f(x,y)]\} + \{cot^2\alpha cosec^2\alpha$ $[(xq+py)^2 + 2i(xp+yq)sin\alpha + 2xypq]FRFT[f(x,y)] - cose$ $[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2\alpha + i(x^2+y^2)cose$ $FRFT[f(x,y)] + x^2y^2cosec^4\alpha FRFT[f(x,y)]$
3	$\{FRFT[xyf(x,y)]\}(p,q)$	$-\left(pqcos^2\alpha + sin^2\alpha\frac{\partial^2}{\partial p\partial q}\right)\{FRFT[f(x,y)]\}$ $+cos\alpha FRFT\{(yp+xq)f(x,y)\}$
4	$\{FRFT[x^2y^2f(x,y)]\}(p,$	$-cos^2\alpha sin^2\alpha[1 - p^2q^2cot^2\alpha + i(p^2+q^2)cot\alpha]\{FRFT[f($ $-cot^2\alpha[(xq+py)^2 + 2y(xp+yq)sin\alpha + 2xypq]\ \{FRF$ $+cos\alpha sin^2\alpha[2pq(py+xq)cot^2\alpha + 2xy(py+xq)cosec^2$ $+i(x^2+y^2)cosec\alpha]\ \{FRFT[f(x,y)]\} + sin^4\alpha\frac{\partial^4}{\partial p^2\partial q^2}\{F$ (p,q)

Conclusion – In this paper we have prove some important operators are applicable while solving heat equation, wave equation, Schrodinger equation, etc.

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