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An Application of Distributional Fourier-Stieltjes Transform

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Abstract:
Stieltjes transform and Fourier transform are related mathematically, since both of this transform have same application in probability and statistics. Fourier transform can be used in communications, linear system analysis, statistics, quantum physics, optics, solution of partial differential equations and antennas. Also the Stieltjes Transform is applied to obtain new inversion formulas of the Laplace transform, in moment problems and to study statistical properties of many-particle spectra of a wide class of new Gaussian ensembles. Distribution of this Integral Transform is the generalization of that transform.

In the present paper, we have defined a new operator $\wedge_{t,x}$ and its adjoint operator $\wedge_{t,x}^{*}$. Using this operator we have given the solution of equation $P\left(\wedge_{t,x}^{*}\right) u(t,x) = f(t,x)$ and Solution of differential equation $P\left(D_{t,x}\right)u(t,x) = f(t,x)$.

Keywords: Fourier Transform, Stieltjes Transform, Fourier-Stieltjes Transform, Adjoint Operator.

1. Introduction:

The basic idea in distribution theory is to reinterpret functions as linear functional acting on a space of test functions. Distributions are a class of linear functional that map a set of test functions into the set of real numbers. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense [4]. Fourier Transform was developed to provide a link between the time domain and frequency domain for non – periodic waveforms. Fourier transform can be used in communications, linear system analysis, statistics, quantum physics, optics, solution of partial differential equations, antennas, to design electric circuit, to solve Integral and differential equations [5]. The Stieltjes is used to obtain new inversion formulas of the Laplace transform, to discuss moment problems in the semi-infinite interval and to study statistical properties of many-particle spectra of a wide class of new Gaussian ensembles [6].

1.1. The Distribution of Fourier-Stieltjes Transform is defined as-

$$FS\{f(t,x)\} = F(s,y) = \langle f(t,x), e^{-ist}\,(x+y)^{-p}\rangle \quad (1.1.1)$$

Where, for each fixed $t(0 < t < \infty), x(0 < x < \infty)$ the r.h.s. of above equation has same as an application of $f(t,x) \in FS_{\alpha}^{*}$ to $e^{-ist}\,(x+y)^{-p} \in FS_{\alpha}$.

In the given paper, we have defined a new operator $\wedge_{t,x}$ and its adjoint operator $\wedge_{t,x}^{*}$ for the distributional Fourier-Stieltjes transform and using this operator, we have given the solution of equation $P\left(\wedge_{t,x}^{*}\right) u(t,x) = f(t,x)$ and Solution of differential equation $P\left(D_{t,x}\right)u(t,x) = f(t,x)$. Distribution of the Integral Transform is the generalization of that transform and here, we have generalized the Fourier-Stieltjes Transform in Distributional sense using the notation and terminology of A.H. Zemanian[1,2].

2. An application of Distributional Fourier-Stieltjes Transform

The kernel of Fourier-Stieltjes Transform as in (1.1) which can also be arrange as-

$$K(t,x,s,p) = e^{-ist}(x+y)^{-p}$$

Differentiate above w.r.t. t, x, we have-

$$D_t D_x K(t,x,s,p) = (-is)e^{-ist}(-p)(x+y)^{-p-1}$$
$$= isp\; e^{-ist}(x+y)^{-p-1}$$

Multiplying both side by $(x+y)$, we get

$$(x+y)\; D_t D_x K(t,x,s,p) = isp\; e^{-ist}(x+y)^{-p}$$

$$(x+y)\ D_t D_x K(t,x,s,p) = isp\ K(t,x,s,p)$$

We construct an operator-

$$\wedge_{t,x} = (x+y) D_t D_x + isp$$

Where, $D_t = \frac{d}{dt}, D_x = \frac{d}{dx}$

$$\wedge_{t,x} K(t,x,s,p) = [(x+y)D_t D_x + isp] K(t,x,s,p)$$
$$= (x+y)D_t D_x K(t,x,s,p) + isp\ K(t,x,s,p)$$
$$= isp\ K(t,x,s,p) + isp\ K(t,x,s,p)$$

$$\wedge_{t,x} K(t,x,s,p) = 2isp\ K(t,x,s,p)$$

Now Consider-

$\wedge_{t,x} K(t,x,s,p) = (C_0)\ K(t,x,s,p)$ Where, $C_0 = 2isp$

Continuing in this way we get

$$\wedge_{t,x}^2 K(t,x,s,p) = (C_0)^2 K(t,x,s,p)$$
$$\wedge_{t,x}^3 K(t,x,s,p) = (C_0)^3 K(t,x,s,p)$$
$$\wedge_{t,x}^4 K(t,x,s,p) = (C_0)^4 K(t,x,s,p)$$

………… and so on

$$\wedge_{t,x}^k K(t,x,s,p) = (C_0)^k K(t,x,s,p) = (2isp)^k K(t,x,s,p)$$

Since, the operator $\wedge_{t,x}^k K(t,x,s,p) = (2isp)^k K(t,x,s,p)$ is obviously linear and continuous, we have

$$FS\{\wedge_{t,x}^k f(t,x)\} = \langle \wedge_{t,x}^k f(t,x), K(t,x,s,p)\rangle$$
$$= \langle f(t,x), \wedge_{t,x}^k K(t,x,s,p)\rangle$$

$FS\{\wedge_{t,x}^k f(t,x)\} = \langle f(t,x), (2isp)^k K(t,x,s,p)\rangle$ for all $f \in FS_\alpha^*$

3. Adjoint Operator $\wedge_{t,x}^*$

We define an operator $\wedge_{t,x}^* : FS_\alpha^* \to FS_\alpha$ using the relation

$$\langle \wedge_{t,x}^* f(t,x), \emptyset(t,x)\rangle = \langle f(t,x), \wedge_{t,x} \emptyset(t,x)\rangle$$

For all $f \in FS_\alpha^*$ and $\emptyset \in FS_\alpha$. The operator $\wedge_{t,x}^*$ is called the adjoint operator of $\wedge_{t,x}$.

For each $k = 1,2,3, \dots\dots\dots$ we can easily get-

$$\langle (\wedge_{t,x}^*)^k f(t,x), \emptyset(t,x)\rangle = \langle f(t,x), \wedge_{t,x}^k \emptyset(t,x)\rangle$$

It can be really shown that if f is regular distribution generated by an element in FS_α.

Then,

$\wedge_{t,x}^* f = \wedge_{t,x} f,$ for each $k = 1,2,3 \dots\dots$

We have-

$$\langle (\wedge_{t,x}^*)^k f(t,x), K(t,x,s,p)\rangle = \langle f(t,x), \wedge_{t,x}^k K(t,x,s,p)\rangle$$
$$= \langle f(t,x), (2isp)^k K(t,x,s,p)\rangle$$
$$= \langle f(t,x), (C_0)^k K(t,x,s,p)\rangle$$

$$\langle (\wedge_{t,x}^*)^k f(t,x), K(t,x,s,p)\rangle = (C_0)^k \langle f(t,x), K(t,x,s,p)\rangle$$

$FS\left\{(\wedge_{t,x}^*)^k f(t,x)\right\} = (C_0)^k\ FS\{f(t,x)\}$ for all $f \in FS_\alpha^*$

4. An Application of the Fourier-Stieltjes Transform to Differential Equation

4.1. Solution of $P(\wedge_{t,x}^*)\ u(t,x) = f(t,x)$

Consider the Differential equation as

$$P\left(\wedge_{t,x}^{t}\right) u(t,x) = f(t,x) \tag{4.1.1}$$

Where, $f \in FS_{\alpha}^{t}$ and P is any polynomial of degree m. Suppose that the equation (4.1.1) possesses the solution u.

Applying Fourier-Stieltjes Transform to (4.1.1) we get-

$$FS\{P\left(\wedge_{t,x}^{t}\right) u\} = FS\{f\}$$

$$P(2isp)FS\{u(t,x)\} = FS\{f(t,x)\} \quad \{\text{Since by using } \left\{\left(\wedge_{t,x}^{t}\right)^{k} f(t,x)\right\} = (2isp)^{k} FS\{f(t,x)\}\}$$

$$P(C_0)FS\{u\} = FS\{f\} \tag{4.1.2}$$

If we further assume that the polynomial P is such that for $\in > 0$.

$$|P(C_0)| < \in \neq 0 \qquad \text{For } s > 0, p > 0 \tag{4.1.3}$$

Then under this assumption (4.1.2) gives-

$$FS(u) = [P(2isp)]^{-1}\, FS(f) \tag{4.1.4}$$

Applying inversion of Fourier-Stieltjes Transform we get

$$u = FS^{-1}\left\{\frac{FS(f)}{P(2isp)}\right\}$$

$$u = FS^{-1}\left\{\frac{FS(f)}{P(C_0)}\right\} \tag{4.1.5}$$

Hence the proof.

4.2. Solution of differential equation $P\left(D_{t,x}\right)u(t,x) = f(t,x)$

Consider the differential equation

$$P\left(D_{t,x}\right)u(t,x) = f(t,x) \tag{4.2.1}$$

When $f \in FS_{\alpha}^{t}$ and $P\left(D_{t,x}\right) = \sum_{\substack{|\alpha| \le m \\ |\beta| \le n}} a_{\alpha} D_t^{\alpha} a_{\beta} D_x^{\beta}$ is a linear differential operator of order m, n with constant coefficients a_{α}, a_{β} respectively.

Suppose that the equation (4.2.1) possesses a solution u. Applying Fourier-Stieltjes Transform to (4.2.1) and using we get

$$(x+y)^{q} D_t^{l} D_x^{q}\, K(t,x,s,p) = (-is)^{l}(-p)^{q} e^{-ist}(x+y)^{-p}$$
$$= (-is)^{l}(-p)^{q} K(t,x,s,p)$$

$$(x+y)^{q} D_t^{l} D_x^{q}\, K(t,x,s,p) = (-1)^{l+q}\, (is)^{l}(p)^{q}\, K(t,x,s,p)$$

We have

$$FS\{P\left(D_{t,x}\right)u(t,x)\} = FS\{f(t,x)\} \tag{4.2.2}$$

We can reform them to the Fourier-Stieltjes Transform and hence we get

$$P[(x+y)^{q} D_t^{l} D_x^{q}\, K(t,x,s,p)]\; FS(u) = FS(f) \tag{4.2.3}$$

Under the assumption that the polynomial P is such that

$$P[(x+y)^{q} D_t^{l} D_x^{q}\, K(t,x,s,p)] < \epsilon \qquad \text{for } \epsilon > 0 \in R^{n}$$

Using (4.2.3)

$$FS(u) = P[(x+y)^{q} D_t^{l} D_x^{q}\, K(t,x,s,p)]^{-1}\, FS(f)$$

Applying inversion of Fourier-Stieltjes Transform to above equation, we have

$$u = [FS]^{-1}\left\{\frac{FS(f)}{P[(x+y)^{q} D_t^{l} D_x^{q}\, K(t,x,s,p)]}\right\}$$

Hence proved

Conclusion:

In the present paper, we have discussed about the applications of distributional Fourier-Stieltjes Transform and generalized them to distributional sense.

References:

A. H. Zemanian, "Generalized integral transform," Inter science publisher, New York, 1968.

A. H. Zemanian, "Distribution theory and transform analysis," McGraw Hill, New York, 1965.

V.D. Sharma, P.D. Dolas: Representation Theorem for the Distributional Fourier-Stieltjes Transform, International Journal of Scientific and Innovative Mathematical Research, Volume 2, Issue 10, October 2014, PP 811-814.

Adem Kilicman, Mustafa Bayram and Hassan Eltayeb, " Recent Developments in Integral transforms, Special functions, and their extensions to distribution theory," Abstract and Applied Analysis, Vol. 2013, Article ID 379856, pp. 1-2, 2013.

M. C. Anumaka: Analysis and Applications of Laplace /Fourier Transformations in Electric Circuit, IJRRAS , Vol12, Issue2, August 2012.

R.S.Pathak,Lokenath Debnath: Recent Developments on the Stieltjes Transform of Generalized Functions, Int. Jr. of Math. & Math. Sci., Vol. I0 No. 4 (1987)641-670.

J Berian James: Integral Transforms for You &Me, Royal Observatory Edinburgh Institute for Astronomy, March 2008.

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