THE abc-CONJECTURE: A GLIMPSE ON HISTORY, STATEMENT, EXPOSITION, STATUS AND CONSEQUENCES

Uday S. Thool[1], Prajay S. Thul[2] and Kishor D. Patil[3]
1 Govt. Institute of Science, Nagpur (India)
2 K.V. W.C.L. New Majari, Kuchana (India)
3 B. D. College of Engineering, Sewagram, Wardha (India)
Corresponding author Email : uthool64@gmail.com

Abstract:

The various terms in the abc-conjecture are discussed and some known consequences are stated without proof. The topics in support of truthiness of the conjecture are discussed. New formulae are established to generate abc-triples, which may lead to search quality abc-triples. There are several mathematical tools to compute and verify abc-triples.

Keywords:

abc-conjecture, prime numbers, abc-triple, quality.

Introduction:

This is the most interesting and most discussed latest problem in the Number theory. Unlike Fermat's last theorem and Goldbach conjecture the abc-conjecture is the simplest statement, which relates three integral values satisfying certain conditions and found very-very difficult to establish the result. The conjecture has many interesting applications. In 1985, J. Oesterley and D.W. Masser posed abc-Conjecture. After studying Lucien Szpiro's conjecture on elliptic curves Osterley was inspired to formulate abc-conjecture. Then little later, Masser while studying Mason's and Stother's Theorem for polynomials over Z, put forward his abc-conjecture independently in slightly different form. Then mathematicians around the world got attracted towards and since then it never looked back. The abc-conjecture was stated with the three positive integer variables a, b and c with a simple linear equation: a+b=c, with a little restrictions that, "gcd" (a,b)=1 which gives the conjecture its name. It says that the properties of a & b affect the properties of c. Though the equation is based on addition, but the conjecture's observation is more about multiplication. Minhyong Kim, professor at Oxford University says "It really is about something very, very basic, about a tight constraint that relates multiplicative and additive properties of numbers". Peter Sarnak, professor at Princeton University who is a self-described skeptic of the abc-conjecture says "There's very little evidence for it, I'll only believe it when it's proved". If it is true! Mathematicians believe that it would reveal a deep relationship between addition and multiplication that they never knew before. It is so powerful, in fact, that it would automatically unlock many legendary mathematical puzzles, one of these would be Fermat's last theorem. Japanese mathematician Shinichi Mochizuki posted four papers on the Internet in 2012. He claimed that he had proved the abc-conjecture, a famed simple number theory problem that had stumped mathematicians for almost 3-decades. But Mochizuki neither send his work to the Annals of Mathematics nor leave a message on any of the online forums frequented by mathematicians around the world.

Material and Method:

Prerequisite for The abc-conjecture: We start with few definitions. Definition 2.1: (Radical of a number) The greatest square-free factor of an integer n is called as radical of n, we denote it as f(n). We take f(1)=1. Example 1: f(2)=2, f(27)=3, f(12)=6, f(6)=6,f(128)=2, f(0)=f(∞)=∞. For an integer n>0,∃ primes p_i's & integers α_i's≥0 "such that " n=p_1^(α_1)⋯ p_k^(α_k). In this case we have f(n)=p_1·p_2·…·p_k or f(n)=∏_(p|n)▒p, where p is prime factor of n. 2.1 Properties of Radical (without proof). For positive integers m, n; P1: 1≤f(n)≤n. P2: f(n^m)=f(n). P3: f(m·n)≤f(m)·f(n), equality holds only when gcd(m,n)=1. P4: If p is prime number then f(p)=p. Converse is not true. Definition 2.2: (abc-triple) For relatively prime integers a & b if f(abc)<a<b1. Definition 2.4: (Good triple) An abc-triple is said to be good triple, if it's q≥1.4 Corollary 2.1: There are finitely many good quality abc-triples.

Therom 2.1:

If a triple (a,b,c) is an abc-triple and a_n=a^(2n+1),b_n=b^(2n+1) & c_n=a^(2n+1)+b^(2n+1) then (a_n ⟦,b⟧_n,c_n) is also an abc-triple ∀ n=1,2,⋯. Proof: Let (a,b,c) be any abc-triple, ⇒(a,b)=1,a+b=c "and " f(abc)<="" td="" style="box-sizing: border-box;">

Result and Discussion:

The Statement of abc-conjecture: There are number of statements of the abc-conjecture in various forms. For any ϵ>0,∃ only finitely many positive integers a,b and c relatively prime and a+b=c then c>[f(abc)]^(1+ϵ) For every ϵ>0 & for positive integers a,b and c relatively prime ∃ finitely many c such that the inequality c≤⟦c_ϵ [f(abc)]⟧^(1+ϵ) holds for a+b=c. There exists abc-triple (a,b,c), with greatest quality q, such that for any other quality of abc-triple other than (a,b,c) hit, then ≤q. For every ϵ>0,∃ a constant c_ϵ such that for every positive integers a,b and c satisfying a+b=c.

Table: List of few good quality *abc*-triples

Sr.No.	Quality	a	b	c
1	1.6299	2	$3^{10}109$	23^{5}
2	1.6260	11^{2}	$3^{2}5^{6}7^{3}$	$2^{21}23$
3	1.6235	$19\cdot1307$	$7\cdot29^{2}31^{8}$	$2^{8}3^{22}5^{4}$
4	1.5808	283	$5^{11}13^{2}$	$2^{8}3^{8}17^{3}$
5	1.5679	1	$2\cdot3^{7}$	$5^{4}7$
6	1.5471	7^{3}	3^{10}	$2^{11}29$
7	1.5444	$7^{2}41^{2}311^{3}$	$11^{16}13^{2}79$	$2\cdot3^{3}5^{23}953$
8	1.5367	5^{3}	$2^{9}3^{17}13^{2}$	$11^{5}17\cdot31^{3}137$
9	1.5270	$13\cdot19^{6}$	$2^{30}5$	$3^{13}11^{2}31$
10	1.5222	$3^{18}23\cdot2269$	$17^{3}29\cdot31^{8}$	$2^{10}5^{2}7^{15}$
11	1.5094	$13^{10}37^{2}$	$3^{7}19^{5}71^{4}223$	$2^{26}5^{12}1873$
12	1.5033	$2^{7}23^{8}$	$19^{9}857^{2}$	$3^{22}13\cdot47^{2}263$
13	1.5028	239	$5^{8}17^{3}$	$2^{10}37^{4}$
14	1.4976	$5^{2}7937$	7^{13}	$2^{18}3^{7}13^{2}$
15	1.4924	$2^{2}11$	$3^{2}13^{10}17\cdot151\cdot4423$	$5^{9}139^{6}$

Conclusion:

By formulation of abc–triple we have given a new hope of ray to generate quality triples, by making use of existing ones. The most important part of this paper is to answer the question “Why so importance to such a problem which looks so elementary?” The importance is due to its elusive proof. This abc-conjecture though looking so elementary but defying all attempts in fact there are only two attempts since its conception. Another reason for so much importance is its wide range of applications. Moreover it’s a very-very strong tool in number theory, it can settle down number of problems if it is true; 1.

Fermat's Last Theorem has an elementary proof. 2. abc-conjecture and generalized Szpiro's conjecture are equivalent. 3. Original Hall's conjecture can be settled. 4. Stother's theorem, Roth's theorem can be proved. 5. It can be established that Wieferich's primes are finite 6. Waring's problem follows from explicit abc-conjecture by Alan Baker. 7. Thue-Segel-Roth Theorem has been settled by E. Bombieri using abc-conjecture. 8. Voijta's conjecture on consecutive powerful numbers can be proved. The list is long enough…

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