Boundary Value Problem in Spherical Coordinates

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Abstract: In this paper discuss boundary value problems coordinate system of spherical coordinate. The crucial distinction between initial values problems and boundary value problems is that in the former case we are able to start an acceptable solution at its beginning (initial problem). Spherical coordinate (r,θ,Φ) : x = r cosø sinθ, y = sinΦ sinθ, z = r cosθ. Legendre's equation arises naturally when solving some PDE in spherical co-ordinate system. Usually in forms part of the sturm-Liouville problem which requires it to have bounded eigen functions over axed domain. I solve the Dirichlet problem for the Laplace equation in side a sphere with f(θ)= L cos2θ. I find the solution to the dirichlet problem for Laplace equation exterior to a sphere with the initial condition u(L,θ) = Sinθ

Key Words : Dirichlet problems, sturm – Liouville problems, initial condition, exterior of sphere.

Introduction:

The boundary value problems in which the conditions are specified at more than one point. The crucial distinction between initial values problems and boundary value problems is that in the former case we are able to start an acceptable solution at its beginning and just march it along by numerical integration to its end, while in the present case, the boundary condition at the starting point do not determine a unique solution to start with and a random choice among the solution that satisfy these starting boundary conditions is almost certain not to satisfy the boundary condition at the other specified points.

Example is a fourth order system which has the Form

$y^{(4)}(x) + ky(x) = q$, with boundary conditions

$y(0) = y'(0) = 0$ and $y(L) = y''(L) = 0$

The Neumann boundary condition is a type of boundary condition name after Carl Neumann. When imposed on an ordinary or a partial differential equation, it specifies the value that the derivative of a solution is to take on the boundary of the domain.

In mathematics a spherical co-ordinate system is a co-ordinate system for three dimensional spaces where the position of a point is specified by three numbers.

The Legendre's equations is a family of differential equations differ by the parameters in the following form

$$(1-x^2)\, y'' - 2x\, y' + \lambda y = 0 \text{ -------------- } (1)$$

Or

$$\frac{d}{dx}[(1-x2)y^1] + \lambda y = 0 \text{ ------------------- } (2)$$

The Legendre's equation is a liner 2nd order ODE. x = ±1 are two singular point of the ODE by garabedian PR(4)

A solution near the ordinary point x = 0 is a power series

$$y = \sum_{n=-\infty}^{\infty} a_n \qquad a_n = 0, \ \forall n < 0$$

The radius of convergence for the power series is the distance from the centre of the series to the nearest singular point i.e. R=1, substitute the power series into the ODE, we will obtain the recurrence relation.

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n, \quad n = 0, 1, 2 \ldots\ldots\ldots\ldots\ldots$$

The recurrence relation gives two series solutions known as the Legendre's functions where one is an odd function and the other one is an even function by using convergence tests, we can show that the two series are convergence for j x j < 1. However, the series are generally not convergent at x = -1 except

If x = +1

For the case of $\lambda = \ell(\ell + 1)$:

When, $n = \ell$, $a_{n+2} = \frac{n(n+1) - \lambda(\iota+1)}{(n+2)(n+1)} a_n = 0$

$a_{\ell+2} = a_{\ell+4} = a_{\ell+6} = \ldots\ldots\ldots\ldots\ldots = 0$

Thus, one of the series solutions becomes a polynomial after normalization; we obtain the Legendre polynomial of degree ℓ

Observation and discussion:

Boundary value analysis is a software testing technique in which tests are designed to include representatives of boundary values. The idea comes from the boundary given that, we have a set of test vectors to test the system a topology can be defined on that set. Those inputs which belong to the same equivalence class as defined by the equivalence partitioning theory would constitute the basis (topology) by Agmon Shmuel <1> given that the basis sets are neighbors as defined in neighborhood (mathematics), there would exist a boundary between them. The test vectors on either side of the boundary are called boundary values. In practice this would require that the test vectors can be ordered and that the individual parameters follows some kind of order in courant R(1950)<3>.

In plainer English, values on the minimum and maximum edges of an equivalence partition are tested. The values could be input or output ranges of a software component, can also be the internal implementation. Since these boundaries are common locations for errors that result in software faults they are frequently exercised in test cases.

In mathematics, the Dirichlet boundary condition is a type of boundary condition, named after Johann Peter Dirichlet (1805 – 1859). When imposed on an ordinary or a partial differential equation. It specifies the values a solution needs to take on the boundary of the domain. The question of finding solutions to such equations is known as the Dirichlet problem in Taylor, (2011) <5>.

For an ordinary differential equation for instance

$$y'' + y = 0$$

The Dirichlet boundary conditions on the internal [a, b] take the form $Y(a) = \propto$ and $y(b)=\beta$, Wher $\propto$ and B are given numbers.

Caucny boundary conditions can be understood from the theory of second order, ordinary differential equation, where to have a particular solution one has to specify the value of the function and the value of the derivative at a given intial or boundary point i.e

$Y(a) = \propto$ and $y^1(a) = B$

Where a is a boundary or initial point.

Laplacian operator in spherical co-ordinates are -

$$\nabla^2 u - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \emptyset 2}$$

- A solution is the Lengendre polynomial of degree ℓ, $p_\ell(x)$
- The other solution is a series solution known as the Legendre function of the second kind $Q_\ell(x)$.
- $Q_\ell(x)$ coverges on the $-1 < x < 1$ but unbounded in $-1 \le x \le 1$.
- Since $P_\ell(x)$ and $Q_\ell(x)$ are linearly independent, thus the general solution to the ODE is

  $Y = c_1 P_\ell(x) + c_2 Q_\ell(x)$
- Ortyogonality : $\int_{-1}^{1} p_m(x).p_n(x)dx = \frac{2}{2n+1}\delta mn$
- Rodrigues's formula : $p_n(x) = 2^{\frac{1}{n}} n! \frac{d^n}{dx^n}(x^2-1)^n$
- Generating function : $\emptyset(n,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} p_n(x)t^n$
- $p_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}[\emptyset(x,t)]_{t=0}$

The Legendre's series expansions for f(x), $-1 \le x \le 1$ is

$f(x) - \sum_{n=0}^{\infty} c_n p_n(x)$ where the generalized Fourier Coefficient

C_n is $c_n = \frac{(p_n/f)}{IIp_n II^2}$

Spherical Bessel's equation

$x^2y^{11} + 2xy^1 + [x^2 - n(n+1)]\, y = 0$

Two linearly independent solutions are the spherical Bessel's function of the first and second kinds.

$$J_n(n) - \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x) - (-x)^n\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^n \frac{\sin x}{x}$$

$$Y_n(n) = \sqrt{\frac{\pi}{2x}}\, Y_{n+\frac{1}{2}}(x) = (-x)^n\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^n \frac{\cos x}{x}$$

A Dirichlet problem inside a sphere with radial symmetry by Axler S<2>

- n is radial symmetry, i.e. u(r,θ,Ø) = u(r, θ)
- Laplace Equation $\nabla^2 u(u,\theta) = o, o < r < L, o \le \theta \le \pi$
- $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) = 0$
- Boundary conditions

  u(L, θ) = f(θ), o ≤ θ ≤ π
- Separation of variables: u(r, θ) = R(r). θ(θ)
- ODE $r^2R'' + 2rR' - \lambda R - 0, \quad 0 < r < L(\text{Euler } \varepsilon q^4)$

  $\theta'' + (\mathrm{Cot}\theta)\theta' + \lambda\theta = 0, \quad 0 \le \theta < \pi\,(\text{Legendre } \varepsilon q^4)$

Conclusion:

The ODE for θ can be converted to Legendre by making a change of variable s = cos θ.

I solve the dirichlet problem for the Laplace equation inside a sphere with f(θ) = Lcos2 θ.

I find the solution to the Dirichlet problem for Laplace equation exterior to a sphere with the initial condition. U(L, θ) = Sin θ

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