



## SOME RESULTS ON DIFFERENTIAL POLYNOMIALS

**S. M. Pawar**Head, Department of Mathematics, Lal Bahadur Shastri College, Satara  
shankar62pawar@gmail.com**Preliminaries:**

Let  $f(z)$  be a transcendental meromorphic function in the finite complex plane. Let  $P(f)$  denote the homogeneous differential polynomial in  $f$ . As in Hayman [5], such functions will be called differential polynomials in  $f$ . Thus a differential polynomial  $P$  in  $f$  is the sum of a finite number of terms of the form  $a f^{l_0} (f^{(1)})^{l_1} \dots (f^{(m)})^{l_m}$  where  $f^{(1)}, f^{(2)}, \dots$  are successive derivatives of  $f$  and  $l_0, l_1, \dots, l_m$  are non-negative integers. If  $l_0 + l_1 + \dots + l_m = n$  (a fixed positive integer) in every term of  $P$ , then  $P$  is called homogeneous differential polynomial in  $f$  of degree  $n$ . In general, if  $\max(l_0 + l_1 + \dots + l_m) = n$  where the maximum is taken over the term of  $P$ , then  $P$  is said to be a differential polynomial in  $f$  of degree at most  $n$ .

**Definition 1:** If  $f_1, f_2$  are meromorphic functions, we denote by  $S(r; f_1, f_2)$  a function of  $r$  such that

$$S(r; f_1, f_2) = O\left(\sum_{i=1}^2 T(r, f_i)\right)$$

as  $r \rightarrow \infty$  through all values if  $f_i$ 's are of finite orders and outside a set of finite linear measure.

Here we prove the theorems by using following lemmas.

**Lemma 1.** [9] If  $P$  is a homogeneous differential polynomial in  $f$  of degree

$$n \geq 1, \text{ then } m\left(r, \frac{P}{f^n}\right) = S(r, f).$$

**Lemma 2. :** [1] Let  $P$  be a homogeneous differential polynomial in  $f$  of degree  $n$  and suppose that  $P$  does not involve  $f$ . That is,  $P$  is a homogeneous differential polynomial of degree  $n$  in  $f^{(1)}, f^{(2)}, \dots$  with coefficients of the form  $a(z)$ . If  $P$  is not a constant and  $b_1, b_2, \dots, b_q$  are distinct elements of  $\mathbb{C}$  (where  $q$  is any positive integer), then

$$n \sum_{i=1}^q m(r, b_i, f) + N\left(r, \frac{1}{P}\right) \leq T(r, P) + S(r, f)$$

**Lemma 3.** [10] Let  $f_1, f_2$  be two non-constant meromorphic functions such that  $a_1 f_1 + a_2 f_2 \equiv 1$ , where  $a_1, a_2$  are constants. Then for  $i = 1, 2$

$$T(r, f_i) < \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_i) + S(r, f_1, f_2) \quad \dots\dots(1)$$

**Lemma 4 :** [1] Let  $f$  be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) = S(r, f)$$

and let  $P$  be a homogeneous differential polynomial in  $f$ . Suppose that  $P$  is not a constant. Then the order of  $P$  is equal to the order of  $f$  and

$$\bar{N}(r, P) + \bar{N}\left(r, \frac{1}{P}\right) = S(r, P)$$

so that  $\bar{N}\left(r, \frac{1}{P-a}\right) \neq S(r, P)$  and  $\Theta(a, P) = 0$  for all  $a \neq 0, \neq \infty$  and there exists no evB for  $P$  for

distinct zeros in  $\bar{C} - \{0, \infty\}$ .

**A.P. Sing and Dukane** [11] have proved the following result.

**Theorem A.** Let  $f(z)$  be a meromorphic function and  $\pi_n(f)$  be a homogeneous differential Polynomial of degree  $n$ .

Let  $\frac{T(r, \pi_n(f))}{T(r, f)} \rightarrow \alpha$  as  $r \rightarrow \infty$  where  $\alpha \geq n$ , then

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{pmn},$$

where  $m$  is the highest derivative occurring in  $\pi_n(f)$  and  $p$  is the number of terms in  $\pi_n(f)$ . Here we shall prove the following improvement of the above theorem. The result here is independent of the number of terms in  $P$ .

**Theorem 1:** Let  $f(z)$  be a meromorphic function and  $P(f)$  be a homogeneous differential Polynomial of degree  $n$ .

Let  $\frac{T(r, P)}{T(r, f)} \rightarrow \alpha$  as  $r \rightarrow \infty$  where  $\alpha \geq n$ , then

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{mn}, \tag{2}$$

where  $m$  is the highest derivative occurring in  $P(f)$ .

**Proof :** Let

$$\frac{T(r, P)}{T(r, f)} \rightarrow \alpha \text{ as } r \rightarrow \infty \text{ where } \alpha \geq n$$

$$\begin{aligned} \text{Now } m(r, P) &\leq m \left( r, \frac{P}{f^n} \right) + m(r, f^n) \\ &= n m(r, f) + S(r, f) \quad \text{by Lemma 1} \quad \dots(3) \end{aligned}$$

At a pole of  $f$  of order  $P$  which is not a pole of any of the coefficient  $a(z)$  of  $P$ ,  $P$  has a pole of order at most  $pn + mn$ .

So,

$$N(r, P) \leq nN(r, f) + mn \bar{N}(r, f) + S(r, f). \tag{4}$$

From (3) and (4), we get.

$$T(r, P) \leq nT(r, f) + mn \bar{N}(r, f) + S(r, f).$$

Since  $\frac{T(r, P)}{T(r, f)} \rightarrow \alpha$  it follows that

$$\alpha T(r, f) \leq nT(r, f) + mn \bar{N}(r, f) + S(r, f). \tag{5}$$

Dividing (5), by  $T(r, f)$  and taking limit superior, we get

$$\alpha - n \leq \limsup_{r \rightarrow \infty} \frac{nm \bar{N}(r, f)}{T(r, f)} + \limsup_{r \rightarrow \infty} \frac{S(r, f)}{T(r, f)}.$$

$$\text{Thus } \alpha - n \leq nm(1 - \Theta(\infty, f))$$

Consequently,

$$nm\Theta(\infty, f) \leq nm + n - \alpha \text{ and so}$$

$$\Theta(\infty, f) \leq 1 + \frac{1}{m} - \frac{\alpha}{nm}$$

Remark - If  $m = 1$ ,  $n=1$  then  $\Theta(\infty, f) \leq 2 - \alpha$  this is an interesting generalization of theorem 3 of S.K. Singh and V. N. Kulkarni [12].

**Bhoosnurmath** [1] has proved the following result.

**Theorem B.** Let  $f$  be a meromorphic function of finite order. If  $P$  is a homogeneous differential polynomial in  $f$  of degree  $n$  and if  $P$  does not involve  $f$ , then

$$n \sum_{b \in C} \delta(b, f) \leq \delta(O, P) \limsup_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)}$$

and

$$n \sum_{b \in C} \delta(b, f) \leq \Delta(O, P) \liminf_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)}$$

provided that  $P$  does not reduce to a constant.

In view of this, we can obtain the following theorem.

**Theorem 2.** Let  $f$  be a meromorphic function of finite order. If  $P$  is a homogenous differential polynomial in  $f$  of degree  $n$  and if  $P$  does not involve  $f$ , and

$$\liminf_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} = \limsup_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} = \alpha, \text{ then}$$

$$n \sum_{a \in C} \delta(a, f) \leq \alpha \delta(O, P) \tag{6}$$

and  $n \sum_{a \in C} \delta(a, f) \leq \alpha \Delta(O, P) \tag{7}$

provided that  $P$  does not reduce to a constant.

**Corollary 2.2.1 :** Let  $f$  be a meromorphic function of finite order with

$$\lim_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} = \alpha, \sum_{a \in C} \delta(a, f) = 1$$

If  $P$  is a homogeneous differential polynomial in  $f$  of degree  $n$  and not involving  $f$ , then

$$n \leq \alpha \delta(O, P) \text{ and } n \leq \alpha \Delta(O, P) \tag{8}$$

$$\text{If } \alpha = n, \delta(O, P) = \Delta(O, P) = 1 \tag{9}$$

provided that  $P$  does not reduce to a constant. In particular (10) and (11) hold if  $f$  is an entire function

$$\lim_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} \rightarrow \alpha \text{ and } \sum_{a \in C} \delta(a, f) = 1.$$

**Theorem 3:** Let  $P[f]$  be a homogeneous differential polynomial such that each term of  $P$  involves  $f$ , then the order of  $P[f]$  and order of  $f$  are equal.

**Proof.** We have

$$P[f] = \sum a f^{l_0} (f^{(1)})^{l_1} \dots (f^{(m)})^{l_m}$$

$$T(r, P[f]) \leq T(r, a) + l_0 T(r, f) + l_1 T(r, f') + \dots + l_m T(r, f^{(m)}).$$

$$T(r, P[f]) \leq T(r, f) \left( l_0 + l_1 \frac{T(r, f')}{T(r, f)} + l_2 \frac{T(r, f'')}{T(r, f)} + \dots + \frac{T(r, f^{(m)})}{T(r, f)} \right) + S(r, f) \tag{10}$$

we have

$$\limsup_{r \rightarrow \infty} \frac{T(r, P)}{T(r, f)} \leq n [(m+1) - m \Theta(\infty, f)]$$

$$\limsup_{r \rightarrow \infty} \frac{T(r, f^{(m)})}{T(r, f)} \leq m+1, \text{ for all } m \geq 1 \tag{11}$$

Substituting (11) in (10), we get

$$\log T(r, P[f]) \leq \log T(r, f) + \log C$$

$$\rho_p = \limsup_{r \rightarrow \infty} \frac{\log T(r, P(f))}{\log r} \leq \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \rho_f$$

$$\therefore \rho_p \leq \rho_f \tag{12}$$

Since a zero or a pole of  $f$ , which is not a pole of any coefficient  $a(z)$  of  $P$ , is a pole of  $\frac{P}{f^n}$  of degree  $mn$

at most, we have

$$N\left(r, \frac{P}{f^n}\right) \leq mn \left( \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) \right) + S(r, f)$$

$$nT(r, f) = T(r, f^n)$$

$$\begin{aligned} &\leq T\left(r, \frac{f^n}{P}\right) + T(r, P) \\ &\leq T\left(r, \frac{P}{f^n}\right) + T(r, P) + O(1) \\ &\leq T(r, P) + mn\left(\bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f)\right) + S(r, f) \\ &\leq T(r, P) + mn\left(\bar{N}\left(r, \frac{1}{P}\right) + \bar{N}(r, P)\right) \end{aligned}$$

Since each term of P involves f term, it follows that

$$\begin{aligned} \bar{N}(r, f) &\leq N(r, P) \text{ and } \bar{N}\left(r, \frac{1}{f}\right) < \bar{N}\left(r, \frac{1}{P}\right) \\ nT(r, f) &\leq T(r, P) + mn\left(\bar{N}\left(r, \frac{1}{P}\right) + \bar{N}(r, P)\right) + S(r, f) \\ &\leq T(r, P) + mn(2T(r, P)) + S(r, f) \\ &\leq (1 + 2mn)T(r, P) + S(r, f) \\ \rho_f &= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} \leq \limsup_{r \rightarrow \infty} \frac{\log T(r, P[f])}{\log r} = \rho_P \end{aligned} \quad \dots\dots(13)$$

From (12) and (13) we conclude that  $\rho_P = \rho_f$

**Theorem 4 :** Let f be a meromorphic function satisfying

$$\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f}\right) = S(r, f).$$

Let P(f) be a homogeneous differential polynomial which does not reduce to a constant. If G is a meromorphic function such that,

$$\bar{N}\left(r, \frac{1}{G}\right) = S(r, G),$$

then the identity,  $P + G \equiv 1$  is impossible.

This theorem was proved by H.S. Gopalkrishna and S.S. Bhoosnumath [1] but here we improve this theorem removing the condition  $\Theta(0, G) > 0$ . It is also interesting to note that here we use a different technique.

**Proof.** Suppose  $P + G \equiv 1$  holds.

$$\begin{aligned} \text{Now } \bar{N}(r, P) &\leq \bar{N}(r, f) + S(r, f) \\ &= S(r, f), \text{ by hypothesis} \\ \text{Therefore } \bar{N}(r, P) &= S(r, P). \quad \dots\dots(14) \\ \text{Because } S(r, f) &= S(r, P). \\ \bar{N}(r, P) &= S(r, f) = S(r, P). \end{aligned}$$

Also

$$\begin{aligned} \bar{N}\left(r, \frac{1}{P}\right) &\leq (1 + mn)\bar{N}\left(r, \frac{1}{f}\right) + mn\bar{N}(r, f) + S(r, f) \\ &= S(r, f), \text{ by hypothesis} \end{aligned}$$

Therefore

$$\bar{N}\left(r, \frac{1}{P}\right) = S(r, f) = S(r, P) \quad \dots\dots(15)$$

Then by Lemma 3 , we have

$$T(r, P) < \bar{N}\left(r, \frac{1}{P}\right) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, P) + S(r, P, G)$$

$$T(r, P) < \bar{N}\left(r, \frac{1}{G}\right) + S(r, G) + S(r, P, G), \quad \dots(16)$$

by Lemma 4.

Since  $P + G = 1$  holds, it follows that

$$T(r, P) \sim T(r, G) \text{ and } S(r, G) = S(r, P). \quad \dots(17)$$

By (16) and (17), we have

$$T(r, G) < \bar{N}\left(r, \frac{1}{G}\right) + S(r, G) \text{ Or } T(r, G) < S(r, G) \text{ by hypothesis.}$$

This is a contradiction.

Thus,  $P + G \equiv 1$  is impossible.

### References:

**Bhoosnurmath .Subhas .S.**, Ph.D. Thesis. Karnatak University, Dharwad, 1974.

**Gopalkrishna. H. S. and Bhoosnurmath S. S.**, Deficiencies of differential polynomials, *J. Mathematical Science*, Vol-16-18,(1981-83), pp.97-102

**Gopalkrishna H. S. and Bhoosnurmath S. S.**, Exceptional values of differential polynomials, *Math. Cronical*, Vol-8, 1979, pp.73-82.

**Gopalkrishna H. S. and Bhoosnurmath S. S.** On the Deficiencies of Differential Polynomials, *Karnatak University Journal, Science*, Vol-XVIII, 1973, pp.329-335.

**Hayman, W. K. and Miles, J.** On the growth of meromorphic function and its derivatives, *Complex Variables*, 12(1989), 245.

**Hayman, W. K.**, Meromorphic function, *Oxford University Press*, 1964.

**Indrajit Lahiri**, Deficiencies of differential polynomials, *Indian J. pure appl. Math.*, 30 (5): 435-447, May 1999.

**Singh, S. K. and Gopalakrishna, H. S.**, Exceptional values of meromorphic functions, *Math, Ann.*, 191, 1971, pp.121-142.

**Subhas S. Bhoosnurmath and Chhaya M. Hombali, (1998)**: Fix points of certain differential polynomials, *Proc. Ind. Acad. Sci.* Vol. 108, No. 2, June 1998, pp.121-131.

**Indrajit Lahiri**, Uniqueness of meromorphic functions with few poles, *J. Ramanujan Math. Soc.* Vol. -11 No.2 1996. Pp. 175-186.

**Singh, A.P. and Dukane, M.** Phil Dissertation, Shivaji University, Kolhapur 1987.

**Singh S.K. and Kulkarni V.N.**, Characteristic function of meromorphic function and its derivatives; *Annales polonici Mathematici* xxviii (1973). Pp 123-133.