I J R B A T, Issue (XI) Vol (II) May 2023: 412-417

A Double-Blind Peer Reviewed & Refereed Journal

OPEN ACCESS

e-ISSN 2347 – 517X

Original Article

INTERNATIONAL JOURNAL OF RESEARCHES IN BIOSCIENCES, AGRICULTURE AND TECHNOLOGY

OPERATIONAL CALCULUS ON GENERALIZED FRACTIONAL FOURIER-WAVELET TRANSFORM

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Communicated : 27.01.2023 | Revision : 02.03.2023 & 10.03.2023 Accepted : 07.04.2023 | Published : 30.05.2023

ABSTRACT:

Fractional Fourier Transform was first introduced as a way to solve certain classes of ordinary and partial differential equation arising in quantum mechanics. Fractional Fourier Transform has established itself as a powerful tool for the analysis of time varying signals, especially in optics. Wavelet transform is used for decomposing a signal into a set of several basis functions, which are nothing but wavelets. Wavelet transform is a more suitable technique for image compression applications. The aim of this paper is to discuss Operational transform formulae on fractional Fourier-Wavelet transform which are useful to solve some linear and non-linear partial differential equation.

Keywords :- Fractional Fourier Transform, Wavelet Transform, Fractional Fourier-Wavelet Transform, Testing Function Space, Signal Processing.

INTRODUCTION :

Fractional Fourier transform is generalization of Fourier transform which was introduced by Victor Namias in 1980's in the field of quantum mechanics for solving some classes of differential equations efficiently [1]. Later, Ozaktas came up with the discrete Implementation of fractional Fourier transform. Since then, a number of applications fractional Fourier transform have been developed, mostly in the field of optics [2]. The Fourier transform of a function can be considered as a linear differential operator acting on that function. The fractional Fourier transform generalizes this differential operator by letting it depend on a continuous parameter $'\alpha'$, Mathematically, α^{th} power of Fourier transform operator [3].

The Fractional Fourier transform of $f(x)$ is given by

$$FrFT\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, K_\alpha(x,p)\, dx \qquad ...(1.1)$$

where, $K_\alpha(x,p) = C_\alpha e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]}$

where $C_\alpha = \sqrt{\frac{1-icot\alpha}{2\pi}}$.

In 1909 wavelet was firstly introduced by Alfred Haar, the progress in the field of wavelet was relatively slow until 1980's. The wavelet transform has been show to be an appropriate tool for time-frequency analysis. Also it has applications in many fields of signal processing image communication radar etc [5]. The wavelet transform is of interest for the analysis of non-stationary signals because it provides an alternative to classical linear time-frequency representations with better time and frequency localization properties [6]. The continuous wavelet transform stands for a scale-space representation and is an effective way to analyze non-stationary signals and images [6]. The Continuous wavelet transform is a scale-space representation of a given function which may be regarded as a joint representation by identifying the scale with a frequency ratio[7].

http://doi.org/10.29369/ijrbat.2023.02.1.0066

The continuous Wavelet transform of $f(t)$ is given by

$W_{\psi}\{f(t)\}(b,a) = \int_{-\infty}^{\infty} f(t)\bar{\psi}_{b,a}(t)dt \ , \ b \in R, a \in R - \{0\}$ ...(1.2)

where $\psi_{b,a}(t) = \frac{1}{|a|} e^{-i\pi\left(\frac{t-b}{a}\right)^2}$.

The Fractional Fourier-Wavelet transform of $f(x,t)$ is defined as

$FrFWT\{f(x,t)\} = F_{\alpha}(p,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)K_{\alpha}(x,p,t,a,b)dxdt$ ...(1.3)

where

$K_{\alpha}(x,p,t,a,b) = C\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2}$

where $0 < \alpha < \frac{\pi}{2}$ and $C = \sqrt{\frac{1-icot\alpha}{2\pi|a|}}$.

$K_{\alpha}(x,p,t,a,b)$ belongs to the testing function space and $f(x,t)$ lies in its dual space.

The purpose of this paper is briefly introduced fractional Fourier-Wavelet transform and its operational transform formulae like scaling property, Differentiability property, Modulation property, Parseval's identity.

Differential Property:

i) $FrFWT\{f'(x,t)\}(p,a,b) = (-icot\alpha)FrFWT\{xf(x,t)\} + (ipcosec\alpha)FrFWT\{f(x,t)\}$

ii) $FrFWT\{f'(x,t)\}(p,a,b) = \left(-\frac{2i\pi t}{a^2}\right) FrFWT\{tf(x,t)\} + \left(\frac{2i\pi b}{a^2}\right) FrFWT\{f(x,t)\}$.

Proof:

i) Consider

$$FrFWT\{f'(x,t)\}(p,a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{\frac{1-icot\alpha}{2\pi|a|}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f'(x,t)dxdt$$

$$= C\int_{-\infty}^{\infty} e^{i\pi\left(\frac{t-b}{a}\right)^2} \left[\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} f'(x,t)dx\right] dt$$

$$= C\int_{-\infty}^{\infty} e^{i\pi\left(\frac{t-b}{a}\right)^2} \left\{\left(e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} f(x,t)dx\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} \frac{i}{2sin\alpha}[2x\, cos\alpha - 2p]f(x,t)dx\right\} dt$$

$$= C\int_{-\infty}^{\infty} e^{i\pi\left(\frac{t-b}{a}\right)^2} \left\{- \int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} [ix\, cos\alpha - ip\, cosec\alpha]f(x,t)dx\right\} dt$$

$$= (-i\, cos\alpha)C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]}\, e^{i\pi\left(\frac{t-b}{a}\right)^2} xf(x,t)dxdt + (ip\, cosec\alpha)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]}\, e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)dxdt$$

$$FrFWT\{f'(x,t)\}(p,a,b) = (-icot\alpha)FrFWT\{xf(x,t)\} + (ipcosec\alpha)FrFWT\{f(x,t)\}$$

ii) Consider

$$FrFWT\{f'(x,t)\}(p,a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{\frac{1-icot\alpha}{2\pi|a|}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f'(x,t)dxdt$$

$$= C\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]} \left\{\int_{-\infty}^{\infty} e^{i\pi\left(\frac{t-b}{a}\right)^2} f'(x,t)dt\right\} dx$$

http://doi.org/10.29369/ijrbat.2023.02.1.0066

$$= C\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} \left\{\left(e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{i\pi\left(\frac{t-b}{a}\right)^2} \frac{2i\pi(t-b)}{a^2} f(x,t)dt\right\} dx$$

$$= C\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} \left\{-\int_{-\infty}^{\infty} e^{i\pi\left(\frac{t-b}{a}\right)^2} \left(\frac{2i\pi t}{a^2} - \frac{2i\pi b}{a^2}\right) f(x,t)dt\right\} dx$$

$$= \left(-\frac{2i\pi t}{a^2}\right) C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)dtdx + \left(\frac{2i\pi b}{a^2}\right) C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)dtdx$$

$$FrFWT\{f'(x,t)\}(p,a,b) = \left(-\frac{2i\pi t}{a^2}\right) FrFWT\{tf(x,t)\} + \left(\frac{2i\pi b}{a^2}\right) FrFWT\{f(x,t)\}.$$

Modulation Property:

i)

$$FrFWT\{f(x,t)\cos(gx+hy)\}(p,a,b) = \frac{1}{2}\{FrFWT[f(x,t)\, e^{i(gx+hy)}](p,a,b) + FrFWT[f(x,t)\, e^{-i(gx+hy)}](p,a,b)\}$$

ii)

$$FrFWT\{f(x,t)\cos(gx+hy)\}(p,a,b) = \frac{1}{2i}\{FrFWT[f(x,t)\, e^{i(gx+hy)}](p,a,b) - FrFWT[f(x,t)\, e^{-i(gx+hy)}](p,a,b)\}$$

Proof: i) Consider

$$FrFWT\{f(x,t)\cos(gx+hy)\}(p,a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{\frac{1-icot\alpha}{2\pi|a|}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)\cos(gx+ hy)dxdt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)\, \frac{e^{i(gx+hy)}+e^{-i(gx+hy)}}{2} dxdt$$

$$= \frac{1}{2}\left\{C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)\, e^{i(gx+hy)} dxdt + C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t) e^{-i(gx+hy)} dxdt\right\}$$

$$FrFWT\{f(x,t)\cos(gx+hy)\}(p,a,b) = \frac{1}{2}\{FrFWT[f(x,t)\, e^{i(gx+hy)}](p,a,b) + FrFWT[f(x,t)\, e^{-i(gx+hy)}](p,a,b)\}$$

ii) Consider

$$FrFWT\{f(x,t)\sin(gx+hy)\}(p,a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{\frac{1-icot\alpha}{2\pi|a|}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)\sin(gx+hy)dxdt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]} e^{i\pi\left(\frac{t-b}{a}\right)^2} f(x,t)\, \frac{e^{i(gx+hy)}-e^{-i(gx+hy)}}{2i} dxdt$$

http://doi.org/10.29369/ijrbat.2023.02.1.0066

$$=\frac{1}{2i}\left\{C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}f(x,t)\,e^{i(gx+hy)}dxdt - C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}f(x,t)e^{-i(gx+hy)}dxdt\right\}$$

$$FrFWT\{f(x,t)\cos(gx+hy)\}(p,a,b) = \frac{1}{2i}\{FrFWT[f(x,t)\,e^{i(gx+hy)}](p,a,b) - FrFWT[f(x,t)\,e^{-i(gx+hy)}](p,a,b)\}$$

Scaling Property:

$$FrFWT\{f(gx,ht)\}(p,a,b) = \frac{1}{gh}FrFWT\{s(x,t)\}(p,a,b)$$

where $s(x,t) = e^{\frac{i}{2sin\alpha}\left[\left(\frac{1-g^2}{g^2}\right)Q^2cos\alpha-2pQ\left(\frac{1-g}{g}\right)\right]+\frac{i\pi}{a^2}\left(\left(\frac{R}{h}\right)^2-\frac{2Rb}{h}+2Rb-R^2\right)}$

Proof: Consider

$$FrFWT\{f(gx,ht)\}(p,a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\frac{1-icot\alpha}{2\pi|a|}}\,e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}f(gx,ht)dxdt$$

Putting $gx = Q \rightarrow gdx = dQ \rightarrow dx = \frac{dQ}{g}$

$ht = R \rightarrow hdt = dR \rightarrow dt = \frac{dR}{h}$

$$FrFWT\{f(gx,ht)\}(p,a,b) = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}\left[\left(\frac{Q^2}{g^2}+p^2\right)cos\alpha-\frac{2Q}{g}p\right]}e^{\frac{i\pi}{a^2}\left(\frac{R}{h}-b\right)^2}f(Q,R)\frac{dQ}{g}\frac{dR}{h}$$

$$=\frac{C}{gh}e^{\frac{i}{2sin\alpha}p^2cos\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}\left[\left(1+\frac{1-g^2}{g^2}\right)Q^2cos\alpha-\frac{2Q}{g}p+2pQ\left(\frac{1-g}{g}\right)-2pQ\left(\frac{1-g}{g}\right)\right]}e^{\frac{i\pi}{a^2}\left(\left(\frac{R}{h}\right)^2-\frac{2Rb}{h}+b^2\right)}f(Q,R)dQ\,dR$$

$$=\frac{C}{gh}e^{\frac{i}{2sin\alpha}p^2cos\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[Q^2cos\alpha-2Qp]}\,e^{\frac{i}{2sin\alpha}\left(\frac{1-g^2}{g^2}\right)Q^2cos\alpha}\,e^{\frac{-i}{2sin\alpha}\left(2pQ\left(\frac{1-g}{g}\right)\right)}e^{\frac{i\pi}{a^2}\left(\left(\frac{R}{h}\right)^2-\frac{2Rb}{h}+b^2-2Rb+R^2+2Rb-R^2\right)}f(Q,R)dQ\,dR$$

$$=\frac{C}{gh}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(Q^2+p^2)cos\alpha-2Qp]}\,e^{i\pi\left(\frac{R-b}{a}\right)^2}e^{\frac{i}{2sin\alpha}\left[\left(\frac{1-g^2}{g^2}\right)Q^2cos\alpha-2pQ\left(\frac{1-g}{g}\right)\right]+\frac{i\pi}{a^2}\left(\left(\frac{R}{h}\right)^2-\frac{2Rb}{h}+2Rb-R^2\right)}f(Q,R)dQ\,dR$$

$$=\frac{1}{gh}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C\;e^{\frac{i}{2sin\alpha}[(Q^2+p^2)cos\alpha-2Qp]}\,e^{i\pi\left(\frac{R-b}{a}\right)^2}s(x,t)dQdR$$

where $s(x,t) = e^{\frac{i}{2sin\alpha}\left[\left(\frac{1-g^2}{g^2}\right)Q^2cos\alpha-2pQ\left(\frac{1-g}{g}\right)\right]+\frac{i\pi}{a^2}\left(\left(\frac{R}{h}\right)^2-\frac{2Rb}{h}+2Rb-R^2\right)}$

$$FrFWT\{f(gx,ht)\}(p,a,b) = \frac{1}{gh}FrFWT\{s(x,t)\}(p,a,b).$$

http://doi.org/10.29369/ijrbat.2023.02.1.0066

Parseval's Identity:

i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\overline{g(x,t)}dxdt = \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,a,b)\overline{G(p,a,b)}dpdb$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,t)|^2dxdt = \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(p,a,b)|^2dpdb$

Proof:

i) By the definition of fractional Fourier-Wavelet Transform

$$F(p,a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\frac{1-icot\alpha}{2\pi|a|}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}f(x,t)dxdt$$

$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}f(x,t)dxdt$$

where $C = \sqrt{\frac{1-icot\alpha}{2\pi|a|}}$

using inversion formula for fractional Fourier-Wavelet transform

$$g(x,t) = \frac{cosec\alpha}{8a^2C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{-i\pi\left(\frac{t-b}{a}\right)^2}G(p,a,b)dpdb$$

$$\overline{g(x,t)} = \frac{cosec\alpha}{8a^2C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}\overline{G(p,a,b)}dpdb$$

Consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\overline{g(x,t)}dxdt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\left[\frac{cosec\alpha}{8a^2C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}\overline{G(p,a,b)}dpdb\right]dxdt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{cosec\alpha}{8a^2C}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}e^{i\pi\left(\frac{t-b}{a}\right)^2}f(x,t)dxdt\right]\overline{G(p,a,b)}dpdb$$

$$= \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,a,b)\overline{G(p,a,b)}dpdb$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\overline{g(x,t)}dxdt = \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,a,b)\overline{G(p,a,b)}dpdb$$

ii) Let $f(x,t) = g(x,t)$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)\overline{f(x,t)}dxdt = \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(p,a,b)\overline{F(p,a,b)}dpdb$$

$$= \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(p,a,b)|^2dpdb$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,t)|^2dxdt = \frac{cosec\alpha}{8a^2C^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(p,a,b)|^2dpdb$$

http://doi.org/10.29369/ijrbat.2023.02.1.0066

CONCLUSION:

In this paper Scaling property, Differential property, Modulation property and Parseval's identity of fractional Fourier-Wavelet transform are discussed.

REFERENCES:

V. Namias, "The fractional order Fourier transform and its application to quantum mechanics", J. Inst. Maths Applies (1980) 25, 241-265.

H.M. Ozaktes, A. Zalevsky, M.A. Katay, "The FRFT with application in optics and signal processing", John Wiky, Chichester New York, USA, 2001.

R. Sexena, K. Singh, "Fractional Fourier Transform: A Review", IETE Journal of Education, Pages 13-29 | Published online: 25 Jan 2016.

V. D. Sharma, "Operational calculus on generalized two-dimensional fractional Fourier transform", International Journal of Engineering and Innovative Technology (IJEIT) Volume 3, Issue 2, August 2013.

S. M. Debbal, L. Hamza Ch, "Heart sound analysis using the three wavelet transform versions the continuous wavelet transform (CWT), the discrete Wavelet transform (DWT) and the wavelet packet transform (PWT)", Journal of Cardiology Interventions, 1(1), 2021.

A. F. Perez-Rendon, R. Robles, "The Convolution theorem for the continuous wavelet transform" Signal Processing Volume 84, Issue 1, January 2004, Pages 55-67.

G. Kaiser, A friendly guide to wavelets, Birkhauser, Boston, 1994

http://doi.org/10.29369/ijrbat.2023.02.1.0066