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DYNAMICAL ANALYSIS OF THE VAN DER POL EQUATION

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Abstract: In this paper we study the van der Pol equation which exhibits self-excited limit cycle oscillations. A dynamical analysis of van der Pol equation is done by investigating its eigenvalues. Using simulations phase plane and the time evolutions of the variables are plotted for different values of the parameter.
Keywords: Van der Pol equation, Relaxation oscillations, Nullclines, Limit cycles

Introduction:

In 1927, a Dutch electrical engineer Balthasar van der Pol investigated electrical circuits involving vacuum tubes. He found stable oscillations (limit cycles) in it which he called relaxation-oscillations. When the frequency of the driving signal of the circuits are close to that of the limit cycle, the circuit is "entrained" to the driving signal [1]. Van der Pol first qualitatively described the oscillatory behavior of the human heart. To study the human heart dynamics he built a number of electronic circuit models [2]. A pacemaker's working is analogous to his investigations about adding an external driving signal. He tried to find out how to stabilize a heart's irregular beating or "arrhythmias". Van der Pol equation is a prototypical equation for self-excited limit cycle oscillations. It is used to model various biological and physical phenomena. Van der Pol designed an electrical circuit involving a triode valve whose resistive properties change with current. When the current is low, the resistance is negative and it becomes positive as the current increases resulting the system to oscillate [3], [4], [5], [6].

Van der Pol equation:

The van der Pol equation is stated as [3]

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{1}$$

where $\mu \geq 0$ is a parameter. Equation (1) is a simple harmonic oscillator with a nonlinear damping term $\mu(x^2 - 1)\dot{x}$. The damping is positive for $|x| > 1$ and negative for $|x| < 1$. This results large amplitude oscillations to decay but when they become too small they are pumped back up. The energy decreased in one cycle balances the energy pumped in and the system settles into a self-sustained oscillations.

Equation (1) can be written as the system of two first order differential equations as

$$\frac{dx}{dt} = y \, \frac{dy}{dt} = \mu(1 - x^2)y - x \tag{2}$$

The equilibrium point of equations (2) is the origin (0,0).

The Jacobean matrix denoted by J is

$$J = \begin{bmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu(1 - x^2) \end{bmatrix}$$

The Jacobean matrix at the equilibrium point (0,0) is

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

The equilibrium point (0,0) is

- An unstable spiral if $0 < \mu < 2$.
- An unstable node if $\mu > 2$.
- A spiral sink for $-1 \leq \mu < 0$.
- A spiral source for $0 < \mu \leq 1$.

Going back to the van der Pol equation (1), we can write

$$\ddot{x} + \mu(x^2 - 1)\dot{x} = \frac{d}{dt}\left[\dot{x} + \mu\left(\frac{x^3}{3} - x\right)\right]$$

Let $F(x) = \frac{x^3}{3} - x, \qquad w = \dot{x} + \mu F(x)$

Then equation (1) implies that

$$\dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x$$

which may be written as

$$\dot{x} = w - \mu F(x)\dot{w} = -x \tag{3}$$

Further let $y = \frac{w}{\mu}$ then equation (3) becomes

$$\dot{x} = \mu[y - F(x)]\dot{y} = -\frac{1}{\mu}x \tag{4}$$

By Lienards theorem the van der Pol equation has a stable limit cycle for $\mu > 0$.

Consider the cubic nullcline$y = F(x) = \frac{x^3}{3} - x$.

- $y = F(x) = \frac{x^3}{3} - x$ is negative for $0 < x < \sqrt{3}$ and is positive for $x > \sqrt{3}$.
- $F(x)$ is monotonic increasing for $x > 1$.
- The cubic nullcline is of knee shaped and there are four critical points
  $A\left(2, \frac{2}{3}\right)$, $B\left(1, -\frac{2}{3}\right)$, $C\left(-2, -\frac{2}{3}\right)$, $D\left(-1, \frac{2}{3}\right)$.

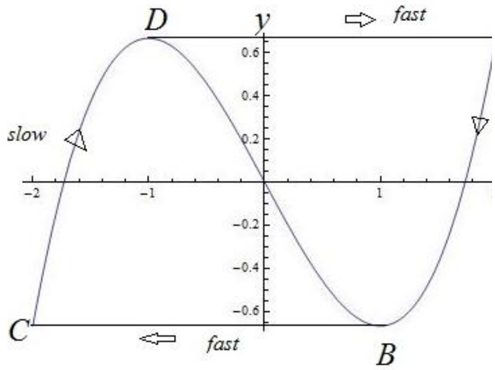
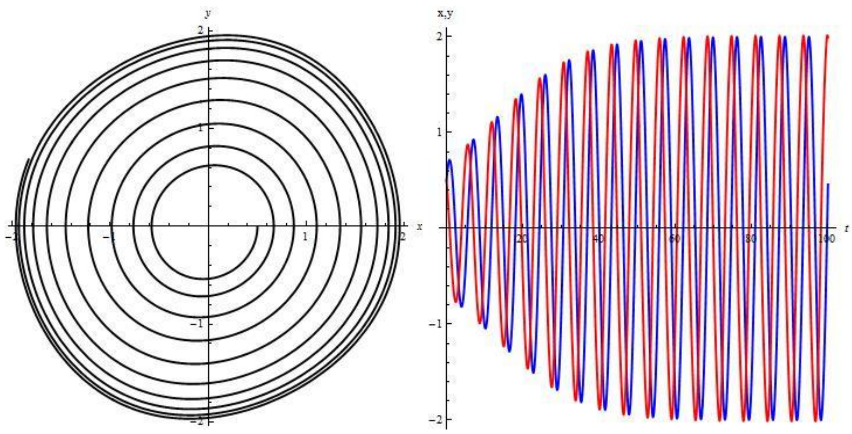


Fig.-1 Nullcline and relaxation oscillations in van der Pol equation (1)

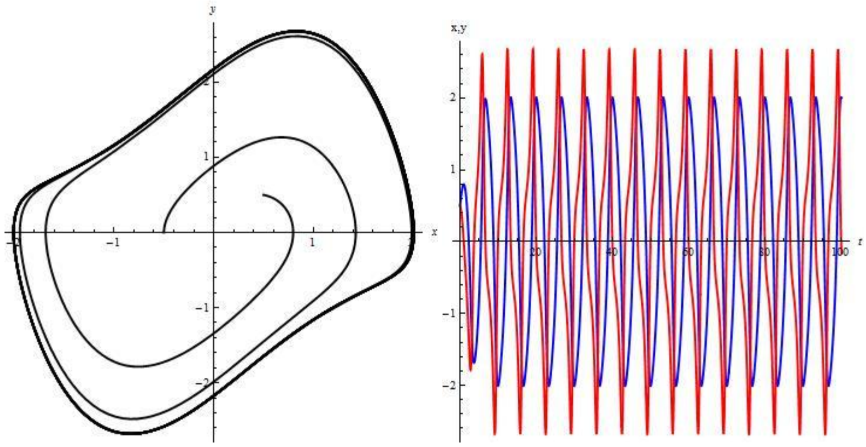
A trajectory starting above the nullclines moves horizontally towards the nullcline. It crosses the nullcline vertically and moves slowly along the backside of the branch AB. When it reaches the point B, it jumps horizontally to the point C and then moves slowly along the branch CD. At D it jumps horizontally to the point A as shown in the Figure1. Thus the system oscillates. Van der Pol termed these oscillations as relaxation oscillations. It has two different time scales $|\dot{y}| \sim o(\mu^{-1}) \ll 1$ and $|\dot{x}| \sim o(\mu) \gg 1$ which operates alternately- a slow buildup which is followed by a fast discharge.

We plot below the phase plane and the variations of x and y with time for the van der Pol equation (1) for different values of μ and initial conditions.



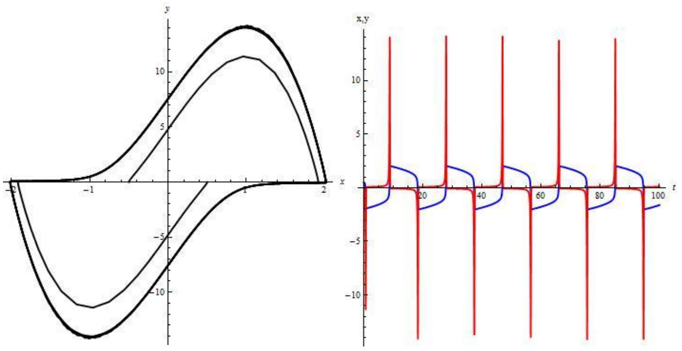
(a) (b)

Fig.-2 (a) Phase plane (b) Variations of x (blue) and y (red) with respect to time of van der Pol equation (1) for $\mu = 0.1$



(a) (b)

Fig.-3 (a) Phase plane (b) Variations of x (blue) and y (red) with respect to time of van der Pol equation (1) for $\mu = 1$



1. (b)

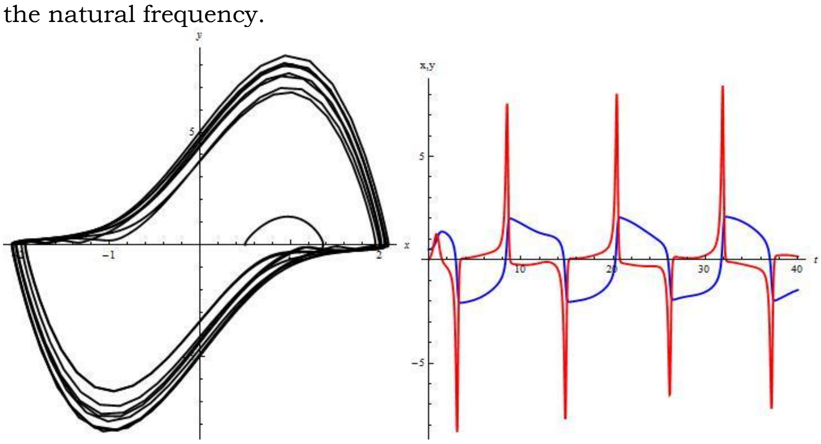
Fig.-3 (a) Phase plane (b) Variations of x (blue) and y (red) with respect to time of van der Pol equation (1) for $\mu = 10$.

Forced van der Pol Equation

Van der Pol and van der Mark [1] introduced a forcing term in equation (1) and they investigated the equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a\ sin\omega t \quad (5)$$

They found the appearance of subharmonical oscillations in the system during changes in the natural frequency.



(a) (b)

Fig.-4 (a) phase portrait and (b) time evolution of x (blue) and that of y (red) with respect to time

In equation (2) an external periodic force $f\,cos\omega t$ is introduced. Figure 4 shows the plots for equation (2) for the values of the parameters $\mu = 5.0, f = 1.0,\ \omega = 1$. A natural periodic stable limit cycle is perturbed by an external periodic force and the resultant periodic oscillations of the combined system are exhibited which is known as mode-locked oscillations. For certain values of the parameters, a period doubling bifurcations and chaos appears in equation (2).

Conclusion:

From the nature of the eigenvalues, it is observed that the van der Pol equation (1) has different types of equilibrium states for different values of the parameter. It has unstable equilibrium point for $0 < \mu < \infty$ and a limit cycle appears in the system. A mode-locked oscillations occur in forced van der Pol equation (2).

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